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Measure-Theoretic Reeb Graphs and Reeb Spaces

Qingsong Wang
University of Utah, Salt Lake City, USA

Guanqun Ma
University of Utah, Salt Lake City, USA

Raghavendra Sridharamurthy
University of Utah, Salt Lake City, USA

Bei Wang
University of Utah, Salt Lake City, USA

Abstract

A Reeb graph is a graphical representation of a scalar function on a topological space that encodes the topology of the level sets. A Reeb space is a generalization of the Reeb graph to a multiparameter function. In this paper, we propose novel constructions of Reeb graphs and Reeb spaces that incorporate the use of a measure. Specifically, we introduce measure-theoretic Reeb graphs and Reeb spaces when the domain or the range is modeled as a metric measure space (i.e., a metric space equipped with a measure). Our main goal is to enhance the robustness of the Reeb graph and Reeb space in representing the topological features of a scalar field while accounting for the distribution of the measure. We first introduce a Reeb graph with local smoothing and prove its stability with respect to the interleaving distance. We then prove the stability of a Reeb graph of a metric measure space with respect to the measure, defined using the distance to a measure or the kernel distance to a measure, respectively.

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1 Introduction

A Reeb graph [48] is a topological descriptor that captures the evolution of level sets of a scalar function. Specifically, given $f : X \to \mathbb{R}$ defined on a topological space X with enough regularity, the Reeb graph of f is a graph where each node corresponds to a critical point of f and each edge captures the relationships among the connected components of the level sets of f. A Reeb space is a generalization of the Reeb graph to a multiparameter function $f : X \to \mathbb{R}^d$. Reeb graphs and Reeb spaces are popular in topological data analysis and visualization; see [12, 38, 60] for surveys.

In this paper, we introduce measure-theoretic Reeb graphs, extensions to the conventional Reeb graph constructions that integrate metric measure spaces—metric spaces endowed with probability measures—to enhance their robustness in capturing the topological features. We argue that a metric measure space arises naturally in data. In many data science applications, we would like to associate weights to data points in the domain or function values in the range, which represent how much we trust these data points or how important their corresponding features are. Conventional Reeb graphs, however, do not take into consideration the data distributions and (possibly) non-uniform importance of data points, leading to discrepancies between the represented and actual topologies of the data. For example, a significant loop in the Reeb graph might be caused by a sparse set of data points or lie in regions of low importance in function values. Our measure-theoretic approach allows Reeb graphs to capture robust topology in data, in line with recent advances in topological data analysis for building robust topological descriptors [13, 47, 19]. Our contributions include:

- We define a Reeb graph of a metric measure space where the domain is equipped with a measure, and present two stability results:
  - We first introduce a Reeb graph with local smoothing (Definition 17) and prove its stability with respect to the interleaving distance (Lemma 18);
  - We then prove the stability of a Reeb graph of a metric measure space with respect to the measure, defined using the distance to a measure [19] and the kernel distance to a measure [47], respectively (Theorem 19 and Theorem 20).
- We expand the measure-theoretic construction to consider a measure on the range, referred to as a range-integrated Reeb graph (Definition 31), and prove its stability (Proposition 33).
- We extend our measure-theoretic constructions (Definition 27 and Definition 35) and stability results to Reeb spaces (Theorem 29, Theorem 30, and Proposition 36).
- We define a geometric notion of interleaving distance between Reeb spaces (Definition 23) that generalizes that of Reeb graphs and prove the stability of Reeb spaces with respect to this interleaving distance (Theorem 26).

2 Related Work

Reeb graphs and Reeb spaces. A Reeb graph [48] is a topological abstraction of the level sets of a scalar function. A Reeb space [35] is analogous to Reeb graphs for a multiparameter function. Theoretical investigations of Reeb graphs, Reeb spaces, and their variants (in particular, Mapper constructions [45]) have been quite active, exploring their distances, information content [33, 23], stability [31, 9, 41, 10, 11, 6, 15, 10, 23], and convergence [5, 44, 33, 21, 16].

There are a number of distances proposed for Reeb graphs and their variants, such as interleaving distance [17, 25, 31, 42, 43, 24], functional distortion distance [9, 11], functional contortion distance [7], edit distance [34, 8, 10, 50], Gromov-Hausdorff distance [22, 55], and bottleneck distance [22]; see [14, 60] for surveys. In particular, de Silva et al. [31] introduced an interleaving distance that quantifies the similarity between Reeb graphs by utilizing a smoothing construction. The smoothing idea was further expanded by Munch and Wang [44], where they proved the convergence between the Reeb space and Mapper [49] in terms of the interleaving distance between their corresponding categorical representations. Bauer et al. [11] showed that the interleaving distance is strongly equivalent to the functional distortion distance [9]. In this paper, we introduce a local smoothing idea and define an interleaving distance between Reeb spaces that generalizes that of Reeb graphs and prove the stability of Reeb spaces with respect to this interleaving distance.

Reeb graphs and their variants have been widely used in data analysis and visualization, including shape analysis [40, 56, 54, 32], flexible isosurfaces [20], isosurface denoising [59], data skeletonization [36], topological quadrangulations [39], loop surgery [53], feature tracking [28], and metric reconstruction of filament structures [27]. See [12, 60] for more applications in computer graphics and data visualization, respectively.

Metric measure spaces. A metric measure space is a metric space equipped with a probability measure, providing a natural framework for statistical inference, machine learning, and data analysis [52]. This concept is particularly relevant in real-world data, often sampled from probabilistic distributions, with inherent distance relationships among data points. In machine learning, metric measure spaces have been used in the study of generative models [4], graph learning [29], and natural language processing [3]. In topological data analysis, metric measure spaces are instrumental in developing statistically robust persistent homology invariants [13, 19], studying functional data [37] and providing measure-theoretic perspective on Vietoris-Rips complexes [1, 2].

Robust geometric inferences. Chazal et al. [26, 19] introduced the distance to a measure function that supports geometric inferences that are robust to noise and outliers. As an alternative method, Phillips et al. [47] showed that robust geometric inference of a point cloud can be achieved by examining its kernel density estimate, and subsequently, the kernel distance. The kernel distance enjoys similar reconstruction properties of distance to a measure, and additionally possesses small coresets [46] for inference tasks. These robust techniques enhance the resilience of geometric inference against noise and outliers, and are utilized in this paper to attune the measures on metric measure spaces.

3 Background on Reeb Graphs and Reeb Spaces

A Reeb graph [48] starts with a topological space X equipped with a continuous real-valued function $f : X \to \mathbb{R}$. It captures the evolution of the level sets of f. Unless otherwise specified, we always work with continuous functions in this paper.

▶ **Definition 1** (Reeb graph). *The Reeb graph is the quotient space $R(X, f) := X/{\sim_f}$ obtained by identifying equivalent points where, for every $x, y \in X, x \sim_f y$ if and only if x and y belong to the same connected component of the level set $f^{-1}(f(x))$.*

By construction, as shown in Figure 1, there is a natural quotient map $\pi : X \to R(X, f)$ that sends a point $x \in X$ to its equivalence class $[x] \in R(X, f)$. Meanwhile, f naturally induces a function $\tilde{f} : R(X, f) \to \mathbb{R}$ defined as $\tilde{f}([x]) = f(x)$. With some appropriate regularity conditions (for example, f being a piecewise linear function on a finite simplicial complex or a Morse function on a compact manifold), the Reeb graph $R(X, f)$ is a finite graph and $\tilde{f}$ is a monotonic function on the edges of $R(X, f)$. The pair (X, f) is referred to as an $\mathbb{R}$-*space* [31]. In this paper, we assume that X and f are regular enough (e.g. constructible $\mathbb{R}$-spaces [31]) so that the Reeb graph $R(X, f)$ is a finite graph. We will use this regularity assumption of Reeb graphs throughout the paper.

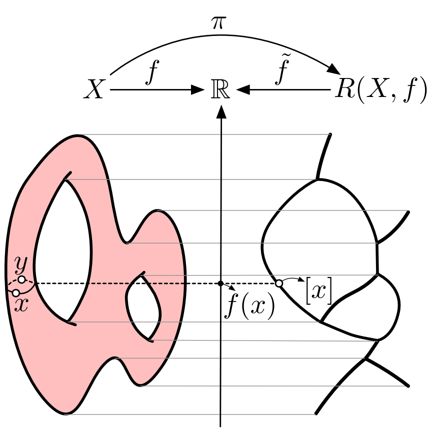


Figure 1 An example of a Reeb graph.

Let (X, f) and (Y, g) be two $\mathbb{R}$-spaces. Following the terminology in [24], we say that a continuous map $\phi : X \to Y$ is a *function preserving map* if $f = g \circ \phi$. A function preserving map $\phi : X \to Y$ induces a map $\tilde{\phi} : R(X, f) \to R(Y, g)$ between the Reeb graphs by sending $[x]$ to $[\phi(x)]$. Additionally, $\tilde{\phi}$ is also a function preserving map between $(R(X, f), \tilde{f})$ and $(R(Y, g), \tilde{g})$. This comes from the universal property of quotient maps; for a proof in the setting of Reeb graphs, see [31, Proposition 2.8].

To simplify the notation, we write a Reeb graph $R(X, f)$ as $G := (G, f)$ with G being a finite graph and f being a real-valued function on G such that f is monotonic on each edge of G. We omit f from (G, f) when it is clear from the context. In particular, G is a special case of an $\mathbb{R}$-space. We say two Reeb graphs are isomorphic if there exist function preserving maps between them that are inverse to each other.

We review the smoothing of Reeb graphs [31] that facilitates the study of the stability of Reeb graphs. It is used to define the interleaving distance between Reeb graphs.

▶ **Definition 2** (Smoothing of Reeb graph [31])**.** *Given a Reeb graph G, the ε-smoothing of G is defined as the Reeb graph of the function:*

$$\begin{aligned} f_\varepsilon : \quad G \times [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R} \\ (x, t) &\longmapsto f(x) + t. \end{aligned}$$

That is, the ε-smoothing of a Reeb graph *is the quotient space $G \times [-\varepsilon, \varepsilon]/\sim_{f_\varepsilon}$, denoted as $S_\varepsilon(G, f)$.*

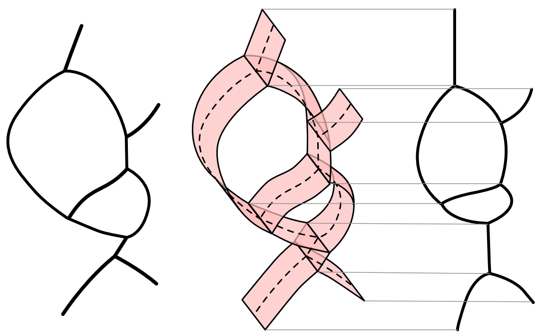


Figure 2 From left to right: a Reeb graph G, its ε-thickening with a function f_ε, and the Reeb graph of the ε-thickening.

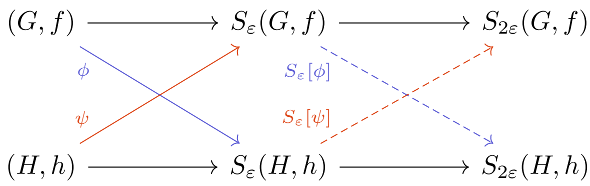
The space $G \times [-\varepsilon, \varepsilon]$ is referred to as the *ε-thickening* of G. Then the ε-smoothing is the Reeb graph of the ε-thickening. See Figure 2 for an example, where the ε-thickening is tilted slightly to reveal its structure. We have the following maps associated with the smoothing of a Reeb graph:

- The zero-section inclusion $\eta : G \to S_\varepsilon(G, f)$ is defined as $\eta(x) = [x, 0]$, where we use $[x, 0]$ to denote the equivalence class of $(x, 0)$ in $S_\varepsilon(G, f)$;
- Let $\phi : (G, f) \to (H, h)$ be a function preserving map between two Reeb graphs. Then we have the induced map $S_\varepsilon[\phi]$ between their smoothings $S_\varepsilon[\phi] : S_\varepsilon(G, f) \to S_\varepsilon(H, h)$ defined as $\phi_\varepsilon([x, t]) = [\phi(x), t]$.

With the above preparations, we can now present the definition of interleaving distance between Reeb graphs introduced by de Silva et al. [31].

▶ **Definition 3** (Interleaving distance [31, Definition 4.35])**.** *For any $\epsilon > 0$, an ε-interleaving between two Reeb graphs (G, f) and (H, h) is a pair of maps, $\phi : (G, f) \to S_\varepsilon(H, h)$ and*

$\psi : (H,h) \to S_\varepsilon(G,f)$ such that the diagram



commutes, where $S_\varepsilon[\phi]$ is the map induced by $\phi : G \times [-\varepsilon,\varepsilon] \to S_\varepsilon(H,h) \times [-\varepsilon,\varepsilon]$ defined as $\phi(x,t) = (\phi(x),t)$. The interleaving distance $d_I((G,f),(H,h))$ is defined as

$$d_I((G,f),(H,h)) = \inf_\varepsilon\{\text{there exists an } \varepsilon\text{-interleaving of } (G,f) \text{ and } (H,h)\}.$$

It is shown in [31] that the interleaving distance is a pseudometric on the set of isomorphism classes of Reeb graphs that takes values in $[0,\infty]$. Additionally, the interleaving distance is zero if and only if the two Reeb graphs are isomorphic.

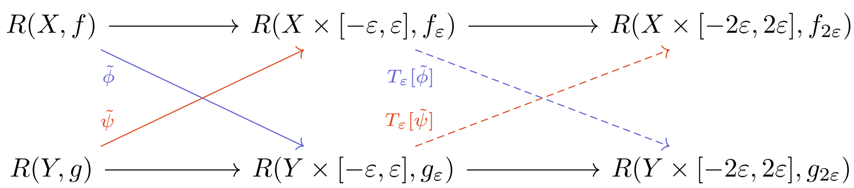
▶ **Proposition 4** ([31, Proposition 4.6]). *Let (G,f) and (H,h) be two Reeb graphs. Then*

$$d_I((G,f),(H,h)) = 0$$

if and only if (G,f) is isomorphic to (H,h).

Note that the smoothing can also be applied to the ambient space directly, that is, we consider $S_\varepsilon(X,f) = X \times [-\varepsilon,\varepsilon]/\sim_{f_\varepsilon}$ where $f_\varepsilon : X \times [-\varepsilon,\varepsilon] \to \mathbb{R}$ is defined as $f_\varepsilon(x,t) = f(x)+t$. Indeed, the above smoothing construction is discussed in [31, Definition 4.19], and this construction is naturally isomorphic to the one used in Definition 2 (in the sense of category theory) as shown in [31, Theorem 4.21]; see also Lemma 24, where we prove this result in the general context of Reeb spaces. This fact allows the following construction of interleaving maps between Reeb graphs.

▶ **Proposition 5.** *Let (X,f) and (Y,g) be two $\mathbb{R}$-spaces. Then $R(X,f)$ and $R(Y,g)$ are ϵ-interleaved if there are function preserving maps $\phi : X \to Y \times [-\varepsilon,\varepsilon]$ and $\psi : Y \to X \times [-\varepsilon,\varepsilon]$ such that the following diagram commutes:*



where $T_\varepsilon[\tilde{\phi}]$ is the map induced by $T_\varepsilon[\phi] : X \times [-\varepsilon,\varepsilon] \to Y \times [-2\varepsilon,2\varepsilon]$ defined as

$$T_\varepsilon[\phi](x,t) = (\mathrm{Pr}_1(\phi(x)), \mathrm{Pr}_2(\phi(x)) + t).$$

We use Pr_1 and Pr_2 to denote the projection maps from $Y \times [-\varepsilon,\varepsilon]$ to Y and $[-\varepsilon,\varepsilon]$ respectively.

Finally, we present the following stability result of Reeb graphs $R(X,f)$ and $R(X,g)$ that are built from the same ambient space X.

▶ **Theorem 6** ([31, Theorem 4.4]). *Let $R(X,f)$ and $R(X,g)$ be two Reeb graphs built from the same ambient space X. Then the interleaving distance defined in Definition 3 satisfies*

$$d_I(R(X,f), R(X,g)) \le \|f-g\|_\infty.$$

The Reeb space [35] is a natural generalization of the Reeb graph to a multiparameter function $f : X \to \mathbb{R}^d$. Again, we will assume that X and f are regular enough (e.g. they induce a constructible cosheaf [30]).

▶ **Definition 7** (Reeb space). *For any continuous $\mathbb{R}^d$-valued functions $f : X \to \mathbb{R}^d$, the Reeb space $R(X, f) := X/{\sim_f}$ is a quotient space of X obtained by identifying points that belong to the same connected component of the level set $f^{-1}(c)$ for any $c \in \mathbb{R}^d$.*

As in the case of the Reeb graph, the multiparameter function f also induces a continuous function $\tilde{f} : R(X, f) \to \mathbb{R}^d$ on the Reeb space $R(X, f)$ by $\tilde{f}([x]) = f(x)$ for any $x \in X$. For two Reeb spaces $R(X, f)$ and $R(Y, g)$, a map $\phi : X \to Y$ is function preserving if $f = g \circ \phi$. Then the function preserving map ϕ induces a map $\tilde{\phi} : R(X, f) \to R(Y, g)$ on the Reeb spaces by $\tilde{\phi}([x]) = [\phi(x)]$ for any $x \in X$. With an abuse of notation, similar to the Reeb graph, we also use the notation (G, f) to denote a Reeb space in Section 6.

4 Background on Measure-Theoretic Concepts

We review measure-theoretic concepts, in particular, the Wasserstein distance between two probability measures on a metric space that originates from optimal transport. We refer the readers to [57] for more details on the Wasserstein distance. We also discuss distance to a measure [26, 19] and kernel distance [51, 47] important for robust structural inference.

▶ **Definition 8** (Metric measure space [52]). *A* metric measure space *is a triple (X, d_X, μ) where (X, d_X) is a metric space and μ is a probability measure on the Borel σ-algebra of X.*

Here, we require that the metric space (X, d_X) is complete and separable, and the measure μ is a locally finite (Borel) probability measure. For simplicity, we use X to denote a metric space (X, d_X), and (X, μ) for a metric measure space, when d_X is obvious from the context.

▶ **Definition 9** (2-Wasserstein distance). *Let (X, d_X) be a metric space and μ, ν be two probability measures on X. The 2-Wasserstein distance between μ and ν is defined as*

$$W_2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} d_X(x, y)^2 d\pi(x, y) \right)^{1/2},$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $X \times X$ with marginals μ and ν.

The distance to a measure function is introduced in [19] and it serves as a robust enhancement for geometric inference.

▶ **Definition 10** (Distance to a measure [19, Definition 1.1]). *Let (X, μ) be a metric measure space and let $m \in (0, 1]$ be a mass parameter. We define the* distance to a measure *function $d_{\mu,m} : X \to \mathbb{R}$ as*

$$d_{\mu,m} : x \in X \mapsto \sqrt{\frac{1}{m} \int_0^m \delta^2_{\mu,s}(x) ds},$$

where $\delta_{\mu,s}$ is defined as $\delta_{\mu,s} : x \in X \mapsto \inf\{r > 0 \mid \mu(\bar{B}(x, r)) > s\}$ and $\bar{B}(x, r)$ denotes the closed ball of radius r centered at x.

The distance to a measure function satisfies the following stability property:

▶ **Theorem 11** ([19, Theorem 3.3] for $\mathbb{R}^n$; [18, Proposition 3.14] for general metric spaces)**.** *Let μ and ν be two probability measures on a metric space (X, d_X) and let $m \in (0,1]$ be a mass parameter. Then: $\|d_{\mu,m} - d_{\nu,m}\|_\infty \leq \frac{1}{\sqrt{m}} W_2(\mu,\nu)$, where $W_2(\mu,\nu)$ is the 2-Wasserstein distance between μ and ν.*

The kernel distance to a measure, as introduced in [47], also offers an alternative robust enhancement for geometric inference. It is closely related to the kernel density estimation from statistics. We generalize this definition from $\mathbb{R}^n$ to general topological spaces by utilizing the notion of *integrally strictly positive definite* kernel functions [51].

▶ **Definition 12** (Integrally strictly positive definite kernel function, [51])**.** *Let X be topological space. A (Borel) measurable function $K : X \times X \to \mathbb{R}$ is called an* integrally strictly positive definite *kernel function if for all finite signed Borel measures μ on X, there is*

$$\int_{X\times X} K(x,x')d\mu(x)d\mu(x') > 0.$$

Examples include the Gaussian kernel function $K(x,x') = \exp(-\|x-x'\|^2/2\sigma^2), \sigma > 0$ on $\mathbb{R}^n$, and certain period function $K(x,x') = \exp^{\alpha\cos(x-x')}\cos(\alpha\sin(x-x')), 0 < \alpha \leq 1$ on the circle $\mathbb{S}^1$ (See Section 3.3 of [51] for details). It is shown in [51] that Defn. 12 allows us to define a metric on the set of probability measures on X.

▶ **Definition 13** (Kernel distance, [51, 47])**.** *Let X be a topological space. Let μ and ν be two probability measures on X. Let K be an integrally strictly positive definite kernel function. Then the kernel distance D_K between μ and ν is defined as*

$$D_K(\mu,\nu) := \sqrt{\kappa(\mu,\mu) + \kappa(\nu,\nu) - 2\kappa(\mu,\nu)},$$

where $\kappa(\mu,\nu)$ is defined as $\kappa(\mu,\nu) := \int_{X\times X} K(x,x')d\mu(x)d\nu(x')$.

▶ **Theorem 14** ([51, Theorem 7])**.** *Let X be a topological space. Let μ and ν be two probability measures on X. Let K be an integrally strictly positive definite kernel function. Then D_K is a metric on the set of probability measures on X.*

The kernel distance (Definition 13) is utilized in [47] to define the kernel distance to a measure by considering the kernel distance between a measure and the Dirac delta measure at a point. We make a slight generalization of the definition to general topological spaces.

▶ **Definition 15** (Kernel distance to a measure, [47])**.** *Let μ be a probability measure on a topological space X. Let K be an integrally strictly positive definite kernel function. Then the kernel distance $D_{\mu,K}$ with respect to μ is a function $D_{\mu,K} : X \to \mathbb{R}$ defined as $D_{\mu,K}(x) = D_K(\mu, \delta_x)$, where δ_x is the Dirac delta measure at x.*

Applying the triangle inequality for the kernel distance, we obtain the following stability result of the kernel distance to a measure function.

▶ **Theorem 16** (Stability of kernel distance to a measure, [47])**.** *Let μ and ν be two probability measures on a topological space X. Let K be an integrally strictly positive definite kernel function. Then $\|D_{\mu,K} - D_{\nu,K}\|_\infty \leq D_K(\mu,\nu)$, where $D_K(\mu,\nu)$ is the kernel distance between μ and ν.*

5 Reeb Graphs for Metric Measure Spaces

With the ingredients from Section 3 and Section 4, we now introduce Reeb graphs for metric measure spaces that are robust to noise in the domain. We achieve this by utilizing the smoothing operation and using either the distance to a measure [19] or the kernel distance to a measure [47] to define a measure-aware local smoothing factor. We first introduce a Reeb graph with local smoothing and prove its stability with respect to the interleaving distance. We then prove the stability of Reeb graphs with respect to the measure, defined using the distance to a measure and the kernel distance to a measure, respectively.

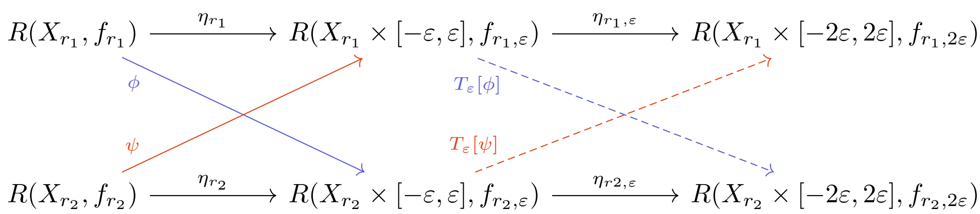
▶ **Definition 17** (Reeb graph with local smoothing). *Let (X, f) be a $\mathbb{R}$-space. Let $r : X \to \mathbb{R}$ be a bounded positive function on X with $M := \sup_{x\in X} r(x)$. The function r is viewed as a local smoothing factor. Let X_r denote the space $X_r = \{(x, t) \in X \times [-M, M] \mid |t| \leq r(x)\}$. Then the function f naturally extends to a function f_r on X_r by $f_r(x, t) = f(x) + t$. We define the* r-smoothed Reeb graph *of (X, f) to be the Reeb graph $R(X_r, f_r)$.*

The standard Reeb graph smoothing is a special case of local smoothing where r is a constant function. The choice of r can be either the distance to measure function $d_{\mu,m}$ or the kernel distance to a measure function $D_{\mu,K}$. We will call them the distance to a measure smoothed Reeb graph and the kernel distance smoothed Reeb graph, denoted as $R(X_{d_{\mu,m}}, f_{d_{\mu,m}})$ and $R(X_{D_{\mu,K}}, f_{D_{\mu,K}})$ respectively.

We have the following stability result regarding the local smoothing of Reeb graphs.

▶ **Lemma 18** (Stability of locally smoothed Reeb graph). *Let X be a topological space and f be a function on X. Additionally, let r_1 and r_2 be two bounded positive function on X with $\varepsilon := \sup_{x\in X} |r_1(x) - r_2(x)|$. Then the r_1-smoothed Reeb graph $R(X_{r_1}, f_{r_1})$ and the r_2-smoothed Reeb graph $R(X_{r_2}, f_{r_2})$ are ε-interleaved.*

Proof. According to Proposition 5, we need to show the existence of maps ϕ and ψ such that the following diagram commutes:



In the above diagram, we use the notation $f_{r_1,\varepsilon}$ to denote the function $X_{r_1} \times [-\varepsilon, \varepsilon] \to \mathbb{R}$ defined as $f_{r_1,\varepsilon}(x, t) = f_{r_1}(x) + t$. We define $f_{r_2,\varepsilon}$, $f_{r_1,2\varepsilon}$, and $f_{r_2,2\varepsilon}$ in a similar manner.

Now, let us introduce the maps ϕ and ψ. We use the pair (x, t) to represent a point in X_{r_1} and the pair $((x, t), t')$ to represent a point in $X_{r_1} \times [-\varepsilon, \varepsilon]$ or $X_{r_1} \times [-2\varepsilon, 2\varepsilon]$.

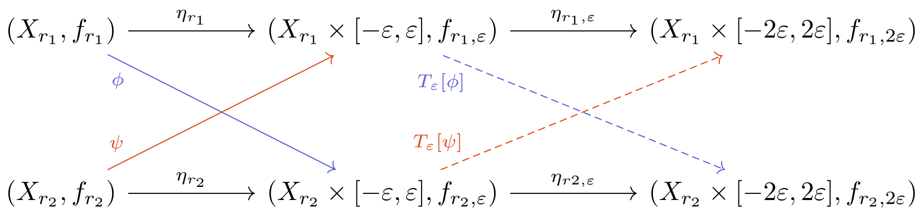
For any $r > 0$, we define the r-parameterized projection map $\pi_r : \mathbb{R} \to [-r, r]$ as

$$\pi_r(t) = \operatorname*{arg\,min}_{-r\leq t'\leq r} |t - t'|.$$

Recall r_1 and r_2 are bounded positive functions on X. We now define the map $\phi : X_{r_1} \to X_{r_2} \times [-\varepsilon, \varepsilon]$ as $\phi : (x, t) \mapsto \left((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t)\right)$. We want to prove that the map ϕ preserves the function value, i.e., $f_{r_1} = f_{r_2,\varepsilon} \circ \phi$ for all $(x, t) \in X_{r_1}$. Indeed, for any $(x, t) \in X_{r_1}$, we have

$$f_{r_2,\varepsilon}(\phi(x, t)) = f_{r_2}\left(x, \pi_{r_2(x)}(t)\right) + t - \pi_{r_2(x)}(t) = f(x) + t = f_{r_1}(x, t).$$

We define the map $\psi : X_{r_2} \to X_{r_1} \times [-\varepsilon, \varepsilon]$ as $\psi : (x,t) \mapsto \big((x, \pi_{r_1(x)}(t)), t - \pi_{r_1(x)}(t)\big)$. By a similar proof as above, we can show that the map ψ preserves the function value, i.e., $f_{r_2} = f_{r_1,\varepsilon} \circ \psi$ for all $(x,t) \in X_{r_2}$. We define η_{r_1} to be the inclusion map $\eta_{r_1} : X_{r_1} \to X_{r_1} \times [-\varepsilon, \varepsilon]$, that is, $\eta_{r_1}(x,t) = ((x,t),0)$. Additionally, let $\eta_{r_1,\varepsilon}$ be the natural inclusion map $X_{r_1} \times [-\varepsilon,\varepsilon] \to X_{r_1} \times [-2\varepsilon, 2\varepsilon]$, and the maps η_{r_2} and $\eta_{r_2,\varepsilon}$ are defined similarly. It is straightforward to see that $\eta_{r_1}, \eta_{r_1,\varepsilon}, \eta_{r_2}, \eta_{r_2,\varepsilon}$ are all function preserving maps. Then we have the following diagram with all the maps preserving function values:



Since each map preserves function values, we obtain a diagram about maps between Reeb graphs induced by the maps between the spaces in the diagram above. To conclude the proof, it suffices to show that the induced diagram between Reeb graphs commutes. We use the notation $[\varphi]$ to denote the induced map between Reeb graphs for any map φ between spaces. By symmetry, it suffices to show

(i) $[T_\varepsilon[\phi]] \circ [\eta_{r_1,\varepsilon}] = [\eta_{r_2,\varepsilon}] \circ [\phi]$, as maps between $R(X_{r_1}, f_{r_1})$ and $R(X_{r_2} \times [-2\varepsilon, 2\varepsilon], f_{r_2,2\varepsilon})$.

(ii) $[T_\varepsilon[\psi]] \circ [\phi] = [\eta_{r_1,\varepsilon}] \circ [\eta_{r_1}]$, as maps between $R(X_{r_1}, f_{r_1})$ and $R(X_{r_1} \times [-2\varepsilon, 2\varepsilon], f_{r_1,2\varepsilon})$.

For item (i), let $(x,t) \in X_{r_1}$, we have

$$
\begin{aligned}
(T_\varepsilon[\phi] \circ \eta_{r_1,\varepsilon})(x,t) &= (T_\varepsilon[\phi])((x,t),0) = (\mathrm{Pr}_1(\phi(x,t)), \mathrm{Pr}_2(\phi(x,t)) + 0) \\
&= \phi((x,t)) = \eta_{r_2,\varepsilon} \circ \phi(x,t),
\end{aligned}
$$

where Pr_1 and Pr_2 are the projection maps from $X_{r_1} \times [-\varepsilon,\varepsilon]$ to X_{r_1} and $[-\varepsilon,\varepsilon]$, respectively. For item (ii), let $(x,t) \in X_{r_1}$, we have

$$
\begin{aligned}
&(T_\varepsilon[\psi] \circ \phi)(x,t) = (T_\varepsilon[\psi])((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t)) \\
&= (\mathrm{Pr}_1(\psi((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t))), \mathrm{Pr}_2(\psi((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t))) + t - \pi_{r_2(x)}(t)).
\end{aligned}
$$

Note that $\psi((x, \pi_{r_2(x)}(t))) = ((x, \pi_{r_1(x)} \circ \pi_{r_2}(t)), \pi_{r_2(x)}(t) - \pi_{r_1(x)} \circ \pi_{r_2}(t))$. Since $|t| \leq r_1(x)$ and $|\pi_{r_2(x)}(t)| \leq t$, $|\pi_{r_2(x)}(t)| \leq r_1(x)$, consequently, $\pi_{r_1(x)} \circ \pi_{r_2}(t) = \pi_{r_2(x)}(t)$. Therefore,

$$
\psi((x, \pi_{r_2(x)}(t))) = ((x, \pi_{r_2(x)}(t)), 0).
$$

Thus, we have

$$
\begin{aligned}
&(T_\varepsilon[\psi] \circ \phi)(x,t) \\
&= (\mathrm{Pr}_1(\psi((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t))), \mathrm{Pr}_2(\psi((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t))) + t - \pi_{r_2(x)}(t)) \\
&= (\mathrm{Pr}_1((x, \pi_{r_2(x)}(t)), 0), \mathrm{Pr}_2((x, \pi_{r_2(x)}(t)), 0) + t - \pi_{r_2(x)}(t)) \\
&= ((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t))
\end{aligned}
$$

Note that $\eta_{r_1,\varepsilon} \circ \eta_{r_1}(x,t) = ((x,t),0)$ and hence the images of $T_\varepsilon[\psi] \circ \phi$ and $\eta_{r_1,\varepsilon} \circ \eta_{r_1}$ are not necessarily the same maps. However, when we pass down to the Reeb graph $R(X_{r_1} \times [-2\varepsilon, 2\varepsilon], f_{r_1,2\varepsilon})$, the induced maps from $(T_\varepsilon[\psi] \circ \phi)$ and $(\eta_{r_1,\varepsilon} \circ \eta_{r_1})$ agree with each other. Indeed, note that the path $\gamma : [0,1] \to X_{r_1} \times [-2\varepsilon, 2\varepsilon]$ defined by

$$
\gamma : s \mapsto ((x, \pi_{r_2(x)}(t) + s(t - \pi_{r_2(x)}(t))), (1-s)t - \pi_{r_2(x)}(t))
$$

satisfies $\gamma(0) = ((x, \pi_{r_2(x)}(t)), t - \pi_{r_2(x)}(t)) = (T_\varepsilon[\psi] \circ \phi)(x,t)$ and $\gamma(1) = ((x,t),0) = (\eta_{r_1,\varepsilon} \circ \eta_{r_1})(x,t)$. Additionally, $f_{r_1,2\varepsilon}$ is a constant function on the path γ and hence $(T_\varepsilon[\psi] \circ \phi)$ and $(\eta_{r_1,\varepsilon} \circ \eta_{r_1})$ are the same maps from $R(X_{r_1}, f_{r_1})$ to $R(X_{r_1} \times [-2\varepsilon, 2\varepsilon], f_{r_1,2\varepsilon})$. This completes the proof. ◀

Now we are ready to prove the stability result of the $d_{\mu,m}$-smoothed Reeb graph and the $D_{\mu,K}$-smoothed Reeb graph with respect to a pair of measures μ and ν; see a full version of the paper [58] for proofs using triangle inequalities.

▶ **Theorem 19** (Stability of $d_{\mu,m}$-smoothed Reeb graph)**.** *Let (X, d_X, μ) and (X, d_X, ν) be two metric measure spaces and $f, g : X \to \mathbb{R}$ be two continuous functions. Let $m \in (0,1]$ be a mass parameter. Then we have*

$$d_I(R(X_{d_{\mu,m}}, f_{d_{\mu,m}}), R(X_{d_{\nu,m}}, g_{d_{\nu,m}})) \leq \|f - g\|_\infty + \frac{1}{\sqrt{m}} W_2(\mu, \nu).$$

Similarly, for a topological space X with an integrally strictly positive definite kernel function K, we can obtain a similar stability result for the $D_{\mu,K}$-smoothed Reeb graph.

▶ **Theorem 20** (Stability of $D_{\mu,K}$-smoothed Reeb graph)**.** *Let X be a topological space. Let μ and ν be two probability measures on X. Let K be an integrally strictly positive definite kernel function on X. Consider two continuous functions $f, g : X \to \mathbb{R}$. Then we have*

$$d_I(R(X_{D_{\mu,K}}, f_{D_{\mu,K}}), R(X_{D_{\nu,K}}, g_{D_{\nu,K}})) \leq \|f - g\|_\infty + D_K(\mu, \nu).$$

▶ **Example 21.** We use an example in Figure 3 to demonstrate the $d_{\mu,m}$-smoothed Reeb graphs (top) and $D_{\mu,K}$-smoothed Reeb graph with Gaussian kernel function (bottom), respectively. Our original space X consists of one large loop containing two small loops α and β, where α is slightly bigger than β. Since the measure based on $D_{\mu,K}$ considers the larger loop α on the bottom left corner to be more important, the $D_{\mu,K}$-smoothed Reeb graph retains α at a larger μ value. On the other hand, the measure based on $d_{\mu,m}$ emphasizes the significance of the smaller loop β on the upper right corner, the $d_{\mu,m}$-smoothed Reeb graph thus retains β at a larger μ value.

6 Reeb Spaces for Metric Measure Spaces

The stability results in Section 5 extends to the setting of Reeb spaces for metric measure spaces. In this section, we use $R(X, f)$ to denote the Reeb space of a multiparameter function $f : X \to \mathbb{R}^d$. We assume that the topological space X and the resulting Reeb space $R(X, f)$ are compact and Hausdorff. The smoothing and hence the notion of interleaving distance of Reeb graph is extended to Reeb space in [44] through categorical language. In this section, we focus on a geometric approach for smoothing a Reeb space. All proofs in this section are given in the full version of the paper [58]. We first introduce the following notations. Let $I_\varepsilon := \{t \in \mathbb{R}^d \mid |t|_\infty \leq \varepsilon\}$ be the ℓ_∞ ball of radius ε centered at the origin, it serves as a higher-dimensional analogue of the 1-dimensional interval.

▶ **Definition 22** (Smoothing of Reeb space)**.** *Let (G, f) be a Reeb space. For any $\varepsilon > 0$, the ε-smoothing $S_\varepsilon(G, f)$ of (G, f) is a Reeb space $R(G \times I_\varepsilon, f_\varepsilon)$ where $f_\varepsilon : G \times I_\varepsilon \to \mathbb{R}^d$ is the continuous function defined by $f_\varepsilon(x,t) = f(x) + t$ for any $(x,t) \in G \times I_\varepsilon$.*

We now define a geometric notion of interleaving distance between Reeb spaces.

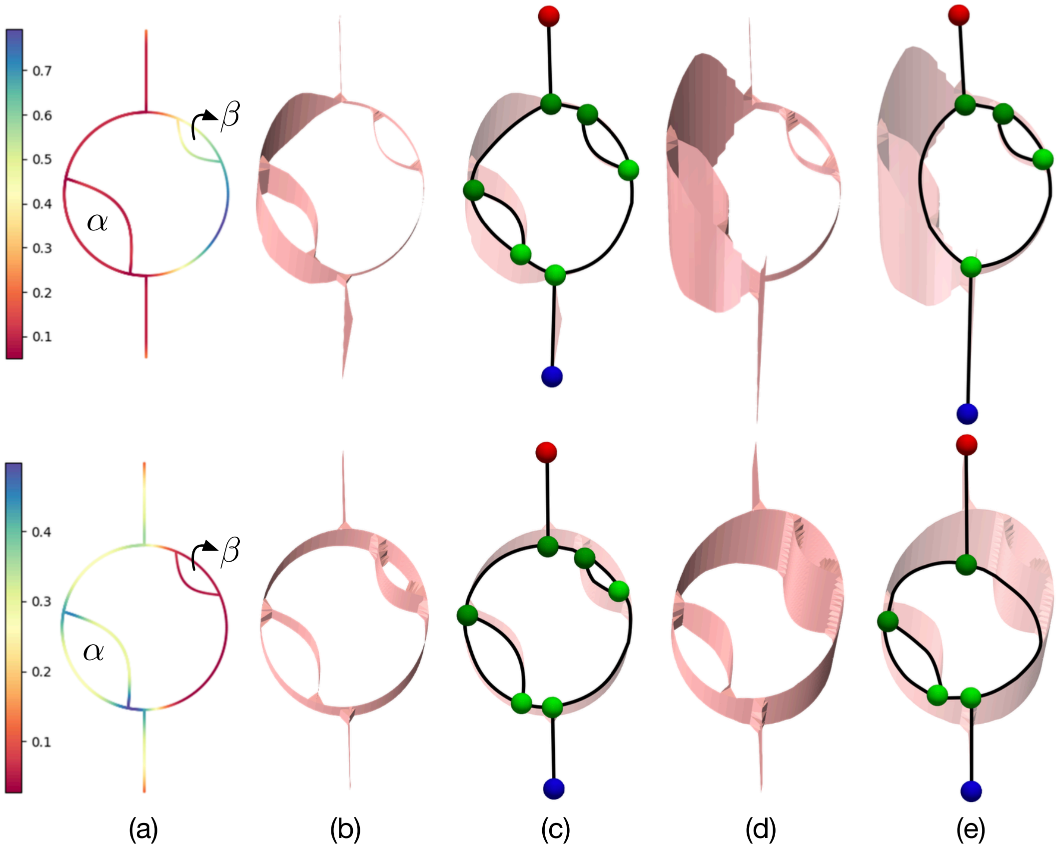


Figure 3 Smoothed Reeb graphs based on distance to a measure (top) and kernel distance to a measure $D_{\mu,K}$ (bottom) with Gaussian kernel function. From left to right: (a) the original topological space X colored by a bounded positive function (e.g., $d_{\mu,m}$ or $D_{\mu,K}$) on X; (b) the (locally) thickened spaces with a small μ value together with (c) the $d_{\mu,m}$-smoothed Reeb graph (top) and the $D_{\mu,K}$-smoothed Reeb graph (bottom); (d)-(e): similar to (b)-(c) with a large μ value.

▶ **Definition 23** (Interleaving distance between Reeb spaces)**.** *For any $\varepsilon > 0$, an ε-interleaving between two Reeb spaces (G, f) and (H, h) is a pair of maps, $\phi : (G, f) \to S_\varepsilon(H, h)$ and $\psi : (H, h) \to S_\varepsilon(G, f)$ such that the diagram*

$$\begin{array}{ccccc} (G,f) & \longrightarrow & S_\varepsilon(G,f) & \longrightarrow & S_{2\varepsilon}(G,f) \\ & {\scriptstyle\phi}\searrow \nearrow{\scriptstyle\psi} & & {\scriptstyle S_\varepsilon[\phi]}\searrow \nearrow{\scriptstyle S_\varepsilon[\psi]} & \\ (H,h) & \longrightarrow & S_\varepsilon(H,h) & \longrightarrow & S_{2\varepsilon}(H,h) \end{array}$$

commutes, where $S_\varepsilon[\phi]$ is the map induced by $\Phi : G \times I_\varepsilon \to S_\varepsilon(H, h) \times I_\varepsilon$ defined as $\Phi(x, t) = (\phi(x), t)$. The interleaving distance $d_I((G, f), (H, h))$ is defined as

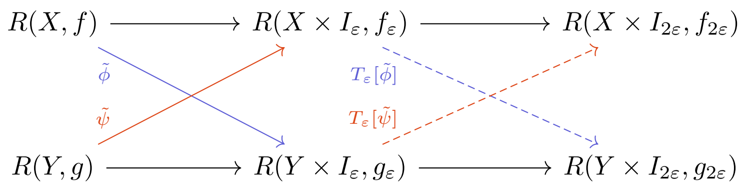
$$d_I((G, f), (H, h)) = \inf_\varepsilon\{\textit{there exists an } \varepsilon\textit{-interleaving of } (G, f) \textit{ and } (H, h)\}.$$

Suppose the Reeb space $R(X, f)$ is induced by a continuous function $f : X \to \mathbb{R}^d$. Then the ε-smoothing $S_\varepsilon(R(X, f), \tilde{f})$ is the same as the Reeb space induced by the continuous function $f_\varepsilon : X \times I_\varepsilon \to \mathbb{R}^d$ on $X \times I_\varepsilon$ defined by $f_\varepsilon(x, t) = f(x) + t$ for any $(x, t) \in X \times I_\varepsilon$. Indeed, we have the following lemma.

▶ **Lemma 24.** *Let $R(X,f)$ be a Reeb space induced by a continuous function $f : X \to \mathbb{R}^d$. Let $R(X \times I_\varepsilon, f_\varepsilon)$ be the Reeb space induced by the continuous function $f_\varepsilon : X \times I_\varepsilon \to \mathbb{R}^d$ on $X \times I_\varepsilon$ defined by $f_\varepsilon(x,t) = f(x) + t$ for any $(x,t) \in X \times I_\varepsilon$. Then there exists a homeomorphism from $S_\varepsilon(R(X,f))$ to $R(X \times I_\varepsilon, f_\varepsilon)$ that preserves function values.*

As a direct consequence, we have the following extension of Proposition 5 to Reeb spaces.

▶ **Proposition 25.** *Let $R(X,f)$ and $R(Y,g)$ be two Reeb spaces induced by continuous functions $f : X \to \mathbb{R}^d$ and $g : Y \to \mathbb{R}^d$ respectively. Then $R(X,f)$ and $R(Y,g)$ are ε-interleaved if there are function preserving maps $\phi : X \to Y \times I_\varepsilon$ and $\psi : Y \to X \times I_\varepsilon$ such that the following diagram commutes:*



where $T_\varepsilon[\tilde{\phi}]$ is the map between Reeb graphs induced by $T_\varepsilon[\phi] : X \times I_\varepsilon \to Y \times I_{2\varepsilon}$ defined as

$$T_\varepsilon[\phi](x,t) = (\mathrm{Pr}_1(\phi(x)), \mathrm{Pr}_2(\phi(x)) + t).$$

Here, we use Pr_1 and Pr_2 to denote the projection maps from $Y \times I_{2\varepsilon}$ to Y and $I_{2\varepsilon}$ respectively.

As in the case of Reeb graph, we have the following stability result for Reeb spaces built from the same space with multiparameter functions; see the full version of the paper [58] for the proof.

▶ **Theorem 26.** *Let $f, g : X \to \mathbb{R}^d$ be two bounded continuous functions on X. Then the Reeb spaces $R(X,f)$ and $R(X,g)$ are $(\|f-g\|_\infty)$-interleaved.*

With the smoothing of Reeb space, we can also define the Reeb space with local smoothing which in turn allows us to define Reeb spaces for metric measure spaces.

▶ **Definition 27** (Reeb space with local smoothing)**.** *Let $f : X \to \mathbb{R}^d$ be a continuous function on X. Additionally, let $r : X \to \mathbb{R}$ be a bounded positive function on X with $M := \sup_{x \in X} r(x)$. The function r is viewed as a local smoothing factor. We use X_r to denote the space $X_r = \{(x,t) \in X \times [-M,M]^d \mid |t| \leq r(x)\}$. Then the function f naturally extends to a function f_r on X_r by $f_r(x,t) = f(x) + t$. We then defined the* r-smoothed Reeb space *of (X,f) to be the Reeb space $R(X_r, f_r)$.*

As in the case of Reeb graph, for a metric measure space (X,d,μ) with $\mathbb{R}^d$-valued function f, we can define the distance to a measure smoothed Reeb graph $R(X_{d_{\mu,m}}, f_{d_{\mu,m}})$ and the kernel distance smoothed Reeb graph $R(X_{D_{\mu,K}}, f_{D_{\mu,K}})$ by using $d_{\mu,m}$ and $D_{\mu,K}$ as the local smoothing factor r in Definition 27.

By considering the variable t belonging to I_ε instead of $t \in [-\varepsilon, \varepsilon]$, the exact same proof of Lemma 18 can be extended to the Reeb space with local smoothing. That is, the Reeb space with local smoothing is stable with respect to the local smoothing factor r. Therefore, we have the following stability result for Reeb space with local smoothing. The proof is identical to the proof of Lemma 18 by simply viewing the parameter t as an element in $\mathbb{R}^d$ instead of $\mathbb{R}$.

▶ **Lemma 28.** *Let (X, d, μ) be a metric measure space and $f : X \to \mathbb{R}^d$ be a continuous function. Let $r_1, r_2 : X \to \mathbb{R}$ be two bounded positive functions on X with $\epsilon := \sup_{x\in X} |r_1(x) - r_2(x)|$. Then the Reeb spaces $R(X_{r_1}, f_{r_1})$ and $R(X_{r_2}, f_{r_2})$ are ϵ-interleaved.*

Likewise, we have the following stability results for Reeb space with local smoothing with respect to functions $d_{\mu,m}$ and $D_{\mu,K}$.

▶ **Theorem 29.** *Let (X, d_X, μ) and (X, d_X, ν) be two metric measure spaces. Let $f, g : X \to \mathbb{R}^d$ be two continuous functions. Let $m \in (0, 1]$ be a mass parameter. Then we have*

$$d_I(R(X_{d_{\mu,m}}, f_{d_{\mu,m}}), R(X_{d_{\nu,m}}, g_{d_{\nu,m}})) \le \|f - g\|_\infty + \frac{1}{\sqrt{m}} W_2(\mu, \nu).$$

▶ **Theorem 30.** *Let (X, d_X, μ) and (X, d_X, ν) be two metric measure spaces. Let K be an integrally strictly positive definite kernel function on X. Let $f, g : X \to \mathbb{R}^d$ be two continuous functions. Let $m \in (0, 1]$ be a mass parameter. Then we have*

$$d_I(R(X_{D_{\mu,K}}, f_{D_{\mu,K}}), R(X_{D_{\nu,K}}, g_{D_{\nu,K}})) \le \|f - g\|_\infty + D_K(\mu, \nu).$$

7 Range-Integrated Reeb Graphs

Our extension of Reeb graphs to metric measure spaces needs not to be limited to measures defined on the domain of the function. We now extend the Reeb graph construction so that it respects a measure μ on the range of a function. For instance, μ may capture the importance of a feature and we would like to understand how μ transforms the shape of a Reeb graph. Let X be a topological space and $f : X \to \mathbb{R}$ be a continuous function. Let μ be a probability measure on $\mathbb{R}$. The *cumulative distribution function* (CDF) of μ is defined as

$$F_\mu(x) := \mu((-\infty, x]) = \int_{-\infty}^{x} d\mu.$$

Therefore, a natural way to adapt the Reeb graph construction when its range comes with a measure μ is to consider the Reeb graph of the function $F_\mu \circ f$. We assume the function $F_\mu \circ f$ is regular so that the Reeb graph $R(X, F_\mu \circ f)$ is a finite graph.

▶ **Definition 31** (Range-integrated Reeb graph)**.** *Let X be a topological space and $f : X \to \mathbb{R}$ be a continuous function. Let μ be a probability measure on $\mathbb{R}$ whose CDF F_μ is continuous. Then the* range-integrated Reeb Graph *of f with respect to μ is defined to be the Reeb graph of $F_\mu \circ f$, denoted as $R(X, F_\mu \circ f)$.*

We provide in Figure 4 an example of the Reeb graph that respects a measure on the range of the function. The intuition behind a range-integrated Reeb graph is that a measure μ on the range enables the vertical scaling (stretching/shrinking) of a Reeb graph according to μ, which subsequently emphasizes certain topological features according to μ.

In the following, we show that the above construction is stable. To this end, we utilize the Kolmogorov-Smirnov distance between two probability measures on $\mathbb{R}$.

▶ **Definition 32** (Kolmogorov-Smirnov distance)**.** *Let μ and ν be two probability measures on $\mathbb{R}$. Then the* Kolmogorov-Smirnov (KS) distance *d_{KS} between μ and ν is defined as*

$$d_{KS}(\mu, \nu) := \sup_{x\in\mathbb{R}} |F_\mu(x) - F_\nu(x)|.$$

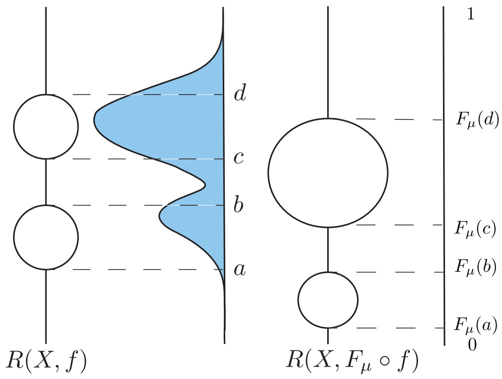


Figure 4 Visualization of a Reeb graph $R(X, f)$ (left) and a range-integrated Reeb Graph $R(X, F_\mu \circ f)$ (right) respectively.

Recall that the Lipschitz constant of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as $\mathrm{Lip}(f) := \sup_{x,y\in\mathbb{R}} \frac{|f(x)-f(y)|}{|x-y|}$. Then we have the following stability result whose proof, as well as other omitted proofs, can be found in the full version of the paper [58].

▶ **Proposition 33.** *Let X be a topological space and $f, g : X \to \mathbb{R}$ two continuous functions. Let μ, ν be two probability measures on $\mathbb{R}$ with continuous CDF F_μ, F_ν respectively. Then we have the following inequality:*

$$d_I(R(X, F_\mu \circ f), R(X, F_\nu \circ g)) \leq \min\{d_{KS}(\mu, \nu) + \mathrm{Lip}(F_\mu)||f - g||_\infty, \\ d_{KS}(\mu, \nu) + \mathrm{Lip}(F_\nu)||f - g||_\infty, 1\}.$$

Specifically, when μ approaches ν and f approaches g, the above inequality implies that the interleaving distance between $R(X, F_\mu \circ f)$ and $R(X, F_\nu \circ g)$ approaches to zero.

Let X be a manifold with a Morse function f. Then the nodes of the Reeb graph $R(X, f)$ are the critical points of f, i.e., the points $x \in X$ such that the gradient $\nabla f(x) = 0$. Under some mild conditions, the range-integrated Reeb graph $R(X, F_\mu \circ f)$ rescales the Reeb graph $R(X, f)$ according to the measure μ on the range of f as in the following proposition.

▶ **Proposition 34.** *Let X be a manifold and $f : X \to \mathbb{R}$ be a Morse function. Let μ be a probability measure on $\mathbb{R}$. If the following conditions hold:*

1. *The measure μ admits a continuously differentiable density function p_μ with respect to the Lebesgue measure λ on $\mathbb{R}$, that is, $\mu(A) = \int_A p_\mu d\lambda$ for any Borel set $A \subset \mathbb{R}$;*
2. *The image of f is contained in the interior of the support of μ, that is, for any $x \in X$, $p_\mu(f(x)) > 0$.*

Then the composition $F_\mu \circ f$ is Morse and the critical points of $F_\mu \circ f$ are the same as the critical points of f with corresponding critical values being $p_\mu(f(x))$ for each critical point x of f. Furthermore, for each critical point x of f, the Hessian of $F_\mu \circ f$ at x has the same number of positive and negative eigenvalues as the Hessian of f at x.

Since the topology of the Reeb graph $R(X, f)$ is determined by the critical points of f and the index of the Hessian of f at each critical point, the above proposition implies that the range-integrated Reeb graph $R(X, F_\mu \circ f)$ maintains the same topology as $R(X, f)$ (under certain conditions) and only stretches/shrinks the Reeb graph $R(X, f)$ according to the measure μ. In Figure 4, we present a visualization of a comparison between the Reeb graph $R(X, f)$ and the range-integrated Reeb graph $R(X, F_\mu \circ f)$ in the setting of Proposition 34.

8 Range-Integrated Reeb Spaces

In this section, we extend the range-integrated Reeb Graph construction to Reeb spaces. Let μ be a probability measure on $\mathbb{R}^d$. For $1 \leq i \leq d$, denote by π_i the projection from $\mathbb{R}^d$ to $\mathbb{R}$ along the i-th coordinate, where $\pi_i(x_1, \ldots, x_d) = x_i$. The marginal distribution of μ along the i-th coordinate, μ_i, is given by $\mu_i(B) = \mu(\pi_i^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$.

▶ **Definition 35** (Range-integrated Reeb space). *Let X be a topological space and $f : X \to \mathbb{R}^d$ be a continuous function. Let μ be a probability measure on $\mathbb{R}^d$ such that the CDF F_{μ_i} of μ_i is continuous for each $1 \leq i \leq d$. We define the coordinate-wise CDF F_μ of μ as follows: $F_\mu : \mathbb{R}^d \to \mathbb{R}^d$ as $F_\mu(x_1, \ldots, x_d) = (F_{\mu_1}(x_1), \ldots, F_{\mu_d}(x_d))$, where F_{μ_i} is the CDF of μ_i. Then, the* range-integrated Reeb space *of f with respect to μ is defined to be the Reeb space of $F_\mu \circ f$, denoted as $R(X, F_\mu \circ f)$.*

Following the same intuition as in the case of range-integrated Reeb graphs, the above construction enables stretching/shrinking of a Reeb space according to a measure μ on the range of a function f. We will show the stability of the range-integrated Reeb space in the following proposition whose proof can be found in the full version of the paper [58].

▶ **Proposition 36.** *Let X be a topological space and $f, g : X \to \mathbb{R}^d$ two continuous functions. Let μ, ν be two probability measures on $\mathbb{R}^d$ such that their coordinate-wise CDFs F_μ, F_ν are continuous. Then we have the following inequality:*

$$d_I(R(X, F_\mu \circ f), R(X, F_\nu \circ g)) \leq \min \Big\{ \mathrm{Lip}(F_\mu)||f - g||_\infty + \max_{1 \leq i \leq d}\{d_{KS}(\mu_i, \nu_i)\}, \\ \mathrm{Lip}(F_\nu)||f - g||_\infty + \max_{1 \leq i \leq d}\{d_{KS}(\mu_i, \nu_i)\}, 1 \Big\}.$$

where the Lipschitz constant of a vector valued fucntion $f : \mathbb{R}^d \to \mathbb{R}^d$ is defined with respect to the ℓ_∞ norm, that is, $\mathrm{Lip}(f) := \sup_{x,y \in \mathbb{R}^d} \frac{\|f(x) - f(y)\|_\infty}{\|x - y\|_\infty}$.

9 Conclusion and Discussion

In this work, we present a novel theoretical framework for Reeb graphs and Reeb spaces, utilizing metric measure spaces in either the domain or the range. Our findings demonstrate the stability of both Reeb graph and Reeb space constructions against perturbations of the function and the measure, thereby offering robust improvements for these topological descriptors. Additionally, as one key component of our framework, we define a geometric notion of interleaving distance between Reeb spaces that generalizes that of Reeb graphs and prove the stability of Reeb spaces with respect to this interleaving distance. Moving forward, we will explore the utility of our framework in topological data analysis and visualization. We will also study the stability of Reeb graphs using distances between their level set persistence diagrams, again in the context of metric measure spaces.

References

1 Michał Adamaszek, Henry Adams, and Florian Frick. Metric reconstruction via optimal transport. *SIAM Journal on Applied Algebra and Geometry*, 2(4):597–619, 2018.

2 Henry Adams, Facundo Mémoli, Michael Moy, and Qingsong Wang. The persistent topology of optimal transport based metric thickenings. *Algebraic & Geometric Topology*, 24(1):393–447, 2024.

3 David Alvarez-Melis and Tommi Jaakkola. Gromov-Wasserstein alignment of word embedding spaces. In Ellen Riloff, David Chiang, Julia Hockenmaier, and Jun'ichi Tsujii, editors, *Proceedings of the 2018 Conference on Empirical Methods in Natural Language Processing*, pages 1881–1890, Brussels, Belgium, 2018. Association for Computational Linguistics.

4 Martin Arjovsky, Soumith Chintala, and Léon Bottou. Wasserstein generative adversarial networks. In *International conference on machine learning*, pages 214–223. PMLR, 2017.

5 Aravindakshan Babu. *Zigzag Coarsenings, Mapper Stability and Gene-network Analyses.* PhD thesis, Stanford University, 2013.

6 Håvard Bakke Bjerkevik. On the stability of interval decomposable persistence modules. *Discrete & Computational Geometry*, 66(1):92–121, 2021.

7 Ulrich Bauer, Håvard Bakke Bjerkevik, and Benedikt Fluhr. Quasi-universality of Reeb graph distances. In *38th International Symposium on Computational Geometry (SoCG 2022)*, volume 224 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 14:1–14:18, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

8 Ulrich Bauer, Barbara Di Fabio, and Claudia Landi. An edit distance for Reeb graphs. In A. Ferreira, A. Giachetti, and D. Giorgi, editors, *Eurographics Workshop on 3D Object Retrieval*, Eindhoven, The Netherlands, 2016. The Eurographics Association.

9 Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. Measuring distance between Reeb graphs. In *Proceedings of the 30th International Symposium on Computational Geometry*, pages 464–474, 2014.

10 Ulrich Bauer, Claudia Landi, and Facundo Memoli. The Reeb graph edit distance is universal. *Foundations of Computational Mathematics*, 21(5):1441–1464, 2020.

11 Ulrich Bauer, Elizabeth Munch, and Yusu Wang. Strong equivalence of the interleaving and functional distortion metrics for Reeb graphs. In *31st International Symposium on Computational Geometry (SoCG 2015)*, volume 34, pages 461–475. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2015.

12 S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Reeb graphs for shape analysis and applications. *Theoretical Computer Science*, 392(1-3):5–22, 2008.

13 Andrew J Blumberg, Itamar Gal, Michael A Mandell, and Matthew Pancia. Robust statistics, hypothesis testing, and confidence intervals for persistent homology on metric measure spaces. *Foundations of Computational Mathematics*, 14:745–789, 2014.

14 Brian Bollen, Erin Chambers, Joshua A. Levine, and Elizabeth Munch. Reeb graph metrics from the ground up. *arXiv preprint arXiv:2110.05631*, 2022.

15 Magnus Botnan and Michael Lesnick. Algebraic stability of zigzag persistence modules. *Algebraic & Geometric Topology*, 18(6):3133–3204, 2018.

16 Adam Brown, Omer Bobrowski, Elizabeth Munch, and Bei Wang. Probabilistic convergence and stability of random Mapper graphs. *Journal of Applied and Computational Topology*, 5:99–140, 2021.

17 Peter Bubenik, Vin de Silva, and Jonathan Scott. Interleaving and Gromov-Hausdorff distance. *arXiv preprint arXiv:1707.06288*, 2018.

18 Mickaël Buchet. *Topological inference from measures.* Theses, Université Paris Sud - Paris XI, December 2014.

19 Mickaël Buchet, Frédéric Chazal, Steve Y Oudot, and Donald R Sheehy. Efficient and robust persistent homology for measures. *Computational Geometry*, 58:70–96, 2016.

20 Hamish Carr, Jack Snoeyink, and Michiel Van De Panne. Flexible isosurfaces: Simplifying and displaying scalar topology using the contour tree. *Computational Geometry*, 43(1):42–58, 2010.

21 Mathieu Carriére, Bertrand Michel, and Steve Oudot. Statistical analysis and parameter selection for Mapper. *Journal of Machine Learning Research*, 19:1–39, 2018.

22 Mathieu Carrière and Steve Oudot. Local equivalence and intrinsic metrics between Reeb graphs. In *33rd International Symposium on Computational Geometry (SoCG 2017)*, volume 77, pages 25:1–25:15, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

23 Mathieu Carriére and Steve Oudot. Structure and stability of the one-dimensional Mapper. *Foundations of Computational Mathematics*, 18(6):1333–1396, 2018.

24 Erin Wolf Chambers, Elizabeth Munch, and Tim Ophelders. A family of metrics from the truncated smoothing of Reeb graphs. In Kevin Buchin and Éric Colin de Verdière, editors, *37th International Symposium on Computational Geometry (SoCG 2021)*, volume 189 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 22:1–22:17, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

25 Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J. Guibas, and Steve Y. Oudot. Proximity of persistence modules and their diagrams. In *25th Annual Symposium on Computational Geometry (SoCG 2009)*, pages 237–246, New York, NY, USA, 2009. Association for Computing Machinery.

26 Frédéric Chazal, David Cohen-Steiner, and Quentin Mérigot. Geometric inference for probability measures. *Foundations of Computational Mathematics*, 11:733–751, 2011.

27 Frédéric Chazal and Jian Sun. Gromov-hausdorff approximation of filament structure using Reeb-type graph. In *Proceedings of the thirtieth annual symposium on Computational geometry*, pages 491–500, 2014.

28 Fang Chen, Harald Obermaier, Hans Hagen, Bernd Hamann, Julien Tierny, and Valerio Pascucci. Topology analysis of time-dependent multi-fluid data using the Reeb graph. *Computer Aided Geometric Design*, 30(6):557–566, 2013.

29 Samantha Chen, Sunhyuk Lim, Facundo Mémoli, Zhengchao Wan, and Yusu Wang. Weisfeiler-Lehman meets Gromov-Wasserstein. In *International Conference on Machine Learning*, pages 3371–3416. PMLR, 2022.

30 Justin Curry. *Sheaves, Cosheaves and Applications*. PhD thesis, University of Pennsylvania, 2014.

31 Vin De Silva, Elizabeth Munch, and Amit Patel. Categorified Reeb graphs. *Discrete & Computational Geometry*, 55(4):854–906, 2016.

32 Tamal K Dey, Fengtao Fan, and Yusu Wang. An efficient computation of handle and tunnel loops via Reeb graphs. *ACM Transactions on Graphics (TOG)*, 32(4):1–10, 2013.

33 Tamal K. Dey, Facundo Mémoli, and Yusu Wang. Topological analysis of nerves, Reeb spaces, Mappers, and multiscale Mappers. In Boris Aronov and Matthew J. Katz, editors, *33rd International Symposium on Computational Geometry*, volume 77 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 36:1–36:16, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

34 Barbara Di Fabio and Claudia Landi. The edit distance for Reeb graphs of surfaces. *Discrete & Computational Geometry*, 55(2):423–461, 2016.

35 Herbert Edelsbrunner, John Harer, and Amit K Patel. Reeb spaces of piecewise linear mappings. In *Proceedings of the twenty-fourth annual symposium on Computational geometry*, pages 242–250, 2008.

36 Xiaoyin Ge, Issam Safa, Mikhail Belkin, and Yusu Wang. Data skeletonization via Reeb graphs. *Advances in Neural Information Processing Systems*, 24, 2011.

37 Haibin Hang, Facundo Mémoli, and Washington Mio. A topological study of functional data and fréchet functions of metric measure spaces. *Journal of Applied and Computational Topology*, 3(4):359–380, 2019.

38 Christian Heine, Heike Leitte, Mario Hlawitschka, Federico Iuricich, Leila De Floriani, Gerik Scheuermann, Hans Hagen, and Christoph Garth. A survey of topology-based methods in visualization. *Computer Graphics Forum (CGF)*, 35(3):643–667, 2016.

39 Franck Hétroy and Dominique Attali. Topological quadrangulations of closed triangulated surfaces using the Reeb graph. *Graphical Models*, 65(1-3):131–148, 2003.

40 Masaki Hilaga, Yoshihisa Shinagawa, Taku Kohmura, and Tosiyasu L. Kunii. Topology matching for fully automatic similarity estimation of 3D shapes. In *Proceedings of the 28th Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH 2001)*, pages 203–212, 2001.

41 Facundo Mémoli, Osman Berat Okutan, and Qingsong Wang. Metric graph approximations of geodesic spaces. *arXiv preprint arXiv:1809.05566*, 2018.

42 Dmitriy Morozov, Kenes Beketayev, and Gunther Weber. Interleaving distance between merge trees. *Proceedings of Topology-Based Methods in Visualization (TopoInVis)*, 2013.

43 Elizabeth Munch and Anastasios Stefanou. *The ℓ^∞-cophenetic metric for phylogenetic trees as an interleaving distance*, volume 17 of *Association for Women in Mathematics Series*, pages 109–127. Springer International Publishing, Cham, 2019.

44 Elizabeth Munch and Bei Wang. Convergence between categorical representations of Reeb space and Mapper. In *32nd International Symposium on Computational Geometry (SoCG 2016)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

45 Monica Nicolau, Arnold J. Levine, and Gunnar Carlsson. Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival. *Proceedings of the National Academy of Sciences*, 108(17):7265–7270, 2011.

46 Jeff M. Phillips. ϵ-samples for kernels. In *Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1622–1632, 2013.

47 Jeff M Phillips, Bei Wang, and Yan Zheng. Geometric inference on kernel density estimates. In *Proceedings of the 31st International Symposium on Computational Geometry*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2015.

48 Georges Reeb. Sur les points singuliers d'une forme de pfaff completement integrable ou d'une fonction numerique [on the singular points of a completely integrable pfaff form or of a numerical function]. *Comptes Rendus Acad. Sciences Paris*, 222:847–849, 1946.

49 Gurjeet Singh, Facundo Mémoli, and Gunnar E. Carlsson. Topological methods for the analysis of high dimensional data sets and 3d object recognition. In Mario Botsch, Renato Pajarola, Baoquan Chen, and Matthias Zwicker, editors, *4th Symposium on Point Based Graphics, 2007*, pages 91–100. Eurographics Association, 2007.

50 Raghavendra Sridharamurthy, Talha Bin Masood, Adhitya Kamakshidasan, and Vijay Natarajan. Edit distance between merge trees. *IEEE Transactions on Visualization and Computer Graphics (TVCG)*, 26(3):1518–1531, 2020.

51 Bharath K Sriperumbudur, Arthur Gretton, Kenji Fukumizu, Bernhard Schölkopf, and Gert RG Lanckriet. Hilbert space embeddings and metrics on probability measures. *The Journal of Machine Learning Research*, 11:1517–1561, 2010.

52 Karl-Theodor Sturm. On the geometry of metric measure spaces I. *Acta Mathematica*, 196(1):65–131, 2006.

53 Julien Tierny, Attila Gyulassy, Eddie Simon, and Valerio Pascucci. Loop surgery for volumetric meshes: Reeb graphs reduced to contour trees. *IEEE Transactions on Visualization and Computer Graphics*, 15(6):1177–1184, 2009.

54 Julien Tierny, Jean-Philippe Vandeborre, and Mohamed Daoudi. Partial 3d shape retrieval by Reeb pattern unfolding. *Computer Graphics Forum*, 28(1):41–55, 2009.

55 Elena Farahbakhsh Touli and Yusu Wang. FPT-algorithms for computing Gromov-Hausdorff and interleaving distances between trees. *Proceedings of the 27th Annual European Symposium on Algorithms*, pages 83:1–83:14, 2019.

56 Tony Tung and Francis Schmitt. The augmented multiresolution Reeb graph approach for content-based retrieval of 3d shapes. *International Journal of Shape Modeling*, 11(01):91–120, 2005.

57 Cédric Villani. *Optimal transport: old and new*, volume 338. Springer, 2009.

58 Qingsong Wang, Guanquan Ma, Raghavendra Sridharamurthy, and Bei Wang. Measure theoretic Reeb graphs and Reeb spaces. *arXiv preprint arXiv:2401.06748*, 2024.

59 Zoë Wood, Hugues Hoppe, Mathieu Desbrun, and Peter Schröder. Removing excess topology from isosurfaces. *ACM Transactions on Graphics (TOG)*, 23(2):190–208, 2004.

60 Lin Yan, Talha Bin Masood, Raghavendra Sridharamurthy, Farhan Rasheed, Vijay Natarajan, Ingrid Hotz, and Bei Wang. Scalar field comparison with topological descriptors: properties

and applications for scientific visualization. *Computer Graphics Forum (CGF)*, 40(3):599–633, 2021.