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Labeled Interleaving Distance for Reeb Graphs

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Abstract

We define the labeled interleaving distance for labeled Reeb graphs (where all vertices are labeled from a fixed set). We prove that the (ordinary) interleaving distance between Reeb graphs equals the minimum of the labeled interleaving distance over all labelings. We also show that under mild conditions, the labeled interleaving distance is a metric on the isomorphism classes of Reeb graphs.

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1 Introduction and Background

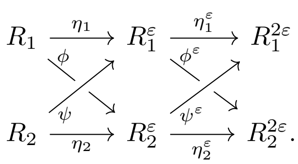
Reeb graphs are topological descriptors that capture topological changes of level sets of scalar fields, with many applications in topological data analysis and visualization [1, 6]. Formally, a Reeb graph, denoted $R = (G, f)$, is a finite multi-graph G equipped with a function $f : G \to \mathbb{R}$ such that the restriction of f on each edge is strictly monotone. It is considered to be a continuous topological space in our setting.

Discriminatory distances for Reeb graphs such as the interleaving distance [2] are often difficult to compute [6]. Recently, Munch and Stefanou [5] introduced a labeled interleaving distance on merge trees with labeled vertices that can be computed in $O(n^2)$ (n being the number of critical points of f). Gasparovic et al. [4] proved that the (ordinary) interleaving distance of merge trees is the minimum of the labeled interleaving distance over all labelings.

In this work, we define a labeled interleaving distance for Reeb graphs, prove that it is a metric on the isomorphism classes of Reeb graphs (under certain conditions on the labelings), and that the (ordinary) interleaving distance is the minimum of the labeled interleaving distance. In particular, our distance can be computed in $O(n^2)$ time for contour trees.

The ordinary interleaving distance is defined using the ε-smoothed Reeb graph [3] R^ε, i.e., the Reeb graph of $G \times [-\varepsilon, \varepsilon]$ together with a function inherited from $\pi : G \times [-\varepsilon, \varepsilon] \to \mathbb{R}$. See Fig. 1 for an example in which R^ε is an ε-smoothed Reeb graph of R.

▶ **Definition 1.** *Let R_1 and R_2 be Reeb graphs. An ε-interleaving between R_1 and R_2 is given by two morphisms, $\phi : R_1 \to R_2^\varepsilon$, $\quad \psi : R_2 \to R_1^\varepsilon$, such that the following diagram commutes,*



The interleaving distance *is defined to be*

$d_I(R_1, R_2) := \inf\{\varepsilon \geq 0 \mid$ *there exists an* ε*-interleaving between* R_1 *and* $R_2\}$.

For a Reeb graph R and its node set V, we define a correspondence $s : V \to R^\varepsilon$ on all nodes in R. Intuitively, s maps a node in R to the point in R^ε where it arrives after moving up (for a split node) or down (for a join node) by ε.

2 Labeled Reeb Graphs and Their Distance

Given a finite label set $L = [N] := \{1, \dots, N\}$ and a Reeb graph $R = (G, f)$ with the node set V, a *labeling* of R is a function $\lambda : L \to V$. We call the triple $R_\lambda = (G, f, \lambda)$ an L*-labeled Reeb graph*. R_λ is *fully-labeled* if λ is surjective. λ is not necessarily injective. A *morphism* between labeled Reeb graphs is defined to be the morphism of the underlying unlabeled Reeb graphs. We want to define a *labeled* ε*-interleaving* between two labeled Reeb graphs by adding label-preserving properties to the *ordinary* ε*-interleaving* defined by the commutative diagram in Def. 1. To do so, we first introduce an ε-path-neighborhood.

▶ **Definition 2.** *Let* $a \in V(R)$ *be a node in the Reeb graph* $R = (G, f)$*. The* ε-path-neighborhood *of* a*, denoted* $P^\varepsilon(a)$*, is* $\pi(\{a\} \times [-\varepsilon, \varepsilon]) \subset R^\varepsilon$ *in the* ε*-smoothed Reeb graph* R^ε*.* $\pi : G \times [-\varepsilon, \varepsilon] \to R^\varepsilon$ *is the quotient map.* $\mathcal{T}^\varepsilon(R)$ *is* $G \times [-\varepsilon, \varepsilon]$ *with the product topology.*

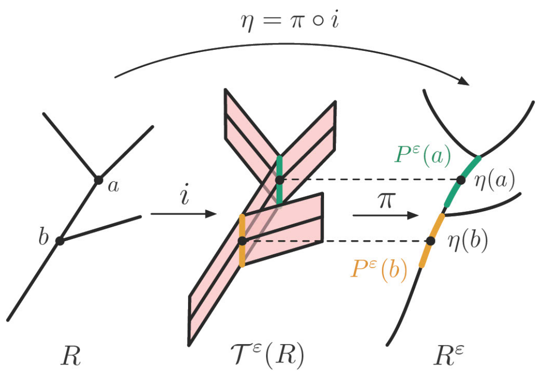


Figure 1 The ε-path-neighborhoods of a and b are highlighted in green and yellow in R^ε.

As illustrated in Fig. 1, we observe that for any point $x \in R$ and any $\varepsilon \geq 0$, $P^\varepsilon(x)$ is a monotonic path in R^ε such that $f^\varepsilon(P^\varepsilon(x)) = [f(x) - \varepsilon, f(x) + \varepsilon)]$.

▶ **Definition 3.** *Let* $R_{1,\lambda_1} = (G_1, f_1, \lambda_1)$ *and* $R_{2,\lambda_2} = (G_2, f_2, \lambda_2)$ *be two* L*-labeled Reeb graphs, and let* $\varepsilon \geq 0$*. We say a pair of morphisms* $\phi : R_1 \to R_2^\varepsilon$ *and* $\psi : R_2 \to R_1^\varepsilon$ *define a* labeled ε-interleaving *between* R_{1,λ_1} *and* R_{2,λ_2} *if the following hold:*

1. ϕ *and* ψ *define an* ε*-interleaving between* R_1 *and* R_2*.*
2. *For each* $\ell \in L$*, we have the following label-preserving properties,*

$$\begin{aligned} &\phi^\varepsilon(s(\lambda_1(\ell))) \in P^\varepsilon(s(\lambda_2(\ell))), \\ &\psi^\varepsilon(s(\lambda_2(\ell))) \in P^\varepsilon(s(\lambda_1(\ell))). \end{aligned} \tag{1}$$

In the above formulae, $P^\varepsilon(s(\lambda_1(\ell))) \subset R_1^{2\varepsilon}$ *and* $P^\varepsilon(s(\lambda_2(\ell))) \subset R_2^{2\varepsilon}$*, since* $s(\lambda_1(l)) \in R_1^\varepsilon$ *and* $s(\lambda_2(l)) \in R_2^\varepsilon$*. The* labeled interleaving distance $d_I^L(R_{1,\lambda_1}, R_{2,\lambda_2})$ *is defined as*

$$\inf\{\varepsilon \geq 0 \mid \textit{there exists a labeled } \varepsilon\textit{-interleaving between } R_{1,\lambda_1} \textit{ and } R_{2,\lambda_2}\}. \tag{2}$$

Fig. 2 illustrates an example for Def. 3 where R_{1,λ_1} and R_{2,λ_2} are *labeled ε-interleaving*. We simplify the notations for node labels. We label nodes $\lambda_1(2)$ with 2, and $s(\lambda_1(2))$ with $s_1(2)$. Nodes with the same function value are assigned the same color. The highlighted areas demonstrate the label-preserving properties in Eqn. (1). It follows from Def. 3 that for all R_1, R_2, λ_1, and λ_2, $d_I^L(R_{1,\lambda_1}, R_{2,\lambda_2}) \geq d_I(R_1, R_2)$. If the label set L is empty, the labeled interleaving distance equals the unlabeled one.

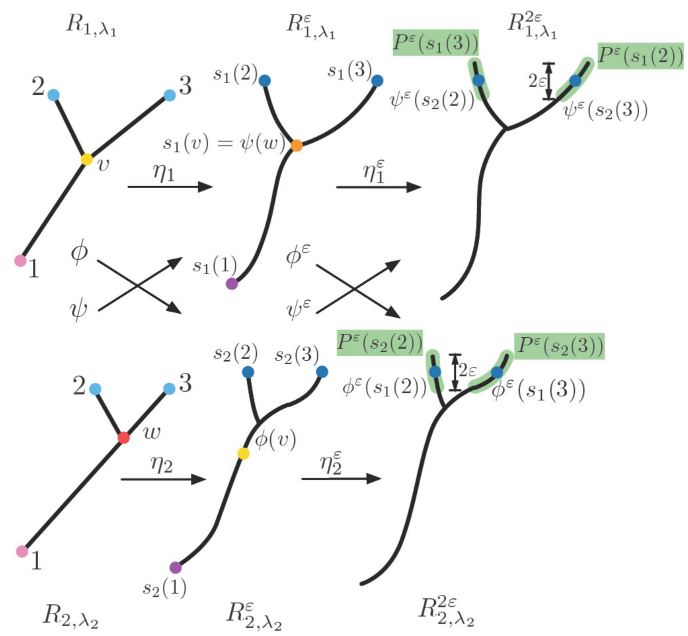


Figure 2 R_{1,λ_1} and R_{2,λ_2} are *labeled ε-interleaving.*

It turns out that the interleaving distance depends only on the topological features significant enough with respect to ε. Therefore, we introduce the ε-essential and ε-inessential nodes. Intuitively, we can disregard nodes connecting to small loops and short edges that do not affect the labeled distance. We call a labeling an *ε-essential labeling* if every ε-essential node is labeled. We consider all possible ε-essential labelings for the following theorem.

▶ **Theorem 4.** *Let R_1, R_2 be Reeb graphs and set $\varepsilon = d_I(R_1, R_2)$. There exist a label set L and ε-essential labelings λ_1 and λ_2 such that for L-labeled Reeb graphs R_{1,λ_1} and R_{2,λ_2},*

$$d_I(R_1, R_2) = d_I^L(R_{1,\lambda_1}, R_{2,\lambda_2}).$$

3 Metric Properties

The labeled interleaving distance can be infinite when two nodes with the same label are originally apart and move in opposite directions when smoothed, see Fig. 3 for an example. For a label set L, we denote the set of all L-labeled Reeb graphs by $\mathcal{R}^L$. We say two labeled Reeb graphs R_{1,λ_1}, R_{2,λ_2} are *consistently labeled* if for each label $\ell \in L$, the nodes $v_1 = \lambda_1(\ell)$ and $v_2 = \lambda_2(\ell)$ move in the same direction when smoothed. A set of labeled Reeb graphs $\mathcal{R} \subset \mathcal{R}^L$ is *consistently labeled* if each pair of labeled Reeb graphs in $\mathcal{R}$ are consistently labeled. Under this condition, the labeled interleaving distance becomes an extended metric.

▶ **Theorem 5.** *Let $\mathcal{R}_c^L$ be a maximal set of isomorphism classes of consistently L-labeled Reeb graphs. The labeled interleaving distance is an extended metric on $\mathcal{R}_c^L$.*

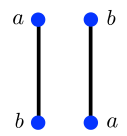


Figure 3 A pair of inconsistently labeled Reeb graphs

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