Finite Element Discretization Strategies for the Inverse Electrocardiographic (ECG) Problem

Dafang Wang, Robert M. Kirby and Chris R. Johnson

Scientific Computing and Imaging Institute, University of Utah, USA

ABSTRACT

Successful employment of numerical techniques for the forward and inverse electrocardiographic (ECG) problems requires the ability to both quantify and minimize approximation errors introduced as part of the discretization process. Conventional finite element discretization and refinement strategies effective for the forward problem may become inappropriate for the inverse problem because of its ill-posed nature. This conjecture leads us to develop discretization strategies specifically for the inverse ECG problem. By quantitatively analyzing the connection between the ill-posedness of the continuum inverse problem and the ill-conditioning of its discretized version, we propose strategies involving hybrid-shaped finite elements to discretize the inverse ECG problem effectively and efficiently. We also propose the criteria for evaluating the quality of the resultant discrete system. The efficacy of the strategies are demonstrated on a realistic torso model in both two and three dimensions.

Index Terms— electrocardiography; inverse problem; finite element method; adaptive refinement; resolution studies

1. INTRODUCTION

Computer modeling of the bioelectric fields in Electrocardiography (ECG) has received considerable interest in recent years because it provides potential clinical tools for noninvasive diagnosis of cardiac diseases (*e.g.* ischemia and arrhythmia), guidance of intervention (*e.g.* ablation and drug delivery), and evaluation of treatments (*e.g.* defibrillation). Both the forward and inverse ECG simulations normally consist of mathematical modeling of the biophysical process as well as geometric approximation of the anatomical human body structures. The modeling equations are then discretized into a numerical system to provide practical solutions. It is vital to assess how accurately numerical techniques solve the mathematical models, the so-called verification process. In order to reduce discretization errors, computational scientists generally use refinement strategies targeted mostly towards well-posed forward problems. However, inverse problems may require different discretization considerations than their corresponding forward problem counterparts. This paper is devoted to the numerical discretization of inverse ECG problems with the finite element method (FEM).

We consider one common formulation of the inverse ECG problem in which one seeks to reconstruct the electric potentials on the heart surface from the body-surface ECG measurements. The mathematical model is given as follows:

$$\begin{aligned} \nabla \cdot (\sigma(\boldsymbol{x}) \nabla u(\boldsymbol{x})) &= 0, \quad \boldsymbol{x} \in \Omega && (1) \\ u(\boldsymbol{x}) &= u_0(\boldsymbol{x}), \ \boldsymbol{x} \in \Gamma_D && (2) \\ \vec{n} \cdot \sigma(\boldsymbol{x}) \nabla u(\boldsymbol{x}) &= 0, \quad \boldsymbol{x} \in \Gamma_N, && (3) \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}), \ \boldsymbol{x} \in \Gamma_N && (4) \end{aligned}$$

where Ω denotes the human torso which is bounded by Γ_D, the epicardial (Dirichlet) boundary, and Γ_N, the torso (Neumann) boundary. The function $u(\boldsymbol{x})$ is the potential field on the domain Ω, $u_0(\boldsymbol{x})$ represents the epicardial potential, and $g(\boldsymbol{x})$ represents the measured body surface potential. $\sigma(\boldsymbol{x})$ is the symmetric positive definite conductivity tensor, and $\vec{n}$ denotes the outward facing vector normal to the torso surface. The forward problem calculates the potential field $u(\boldsymbol{x})$ given the epicardial source $u_0(\boldsymbol{x})$. The inverse problem attempts to recover $u_0(\boldsymbol{x})$ from the measurement $g(\boldsymbol{x})$.

Discretization strategies for the FEM typically either utilize adaptive refinements based on certain element-wise error estimators, or refine regions where the potential field has high spatial gradients [1]. Refinement stategies effective for forward problems may not, however, be appropriate in inverse problems–for example, increasing the numerical resolution beyond a certain level may continue to improve the accuracy of the forward problem, but meanwhile worsens the conditioning of the inverse problem and reduces the solution accuracy.

In contrast to the well-posed forward problem, the inverse problem is often severely ill-conditioned, requiring regularization techniques to guarantee a stable solution. For example, one form of regularization is the discretization itself [2]. We aim to develop discretization strategies that optimally alleviate the ill-conditioning of the inverse ECG problem. The strategies can be used in combination with other classical regularization methods so as to achieve additional improvement of the inverse solution accuracy.

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2. Inverse Problem, Discretization and Ill-posedness

2.1. Finite Element Discretization

The potential field $u(\boldsymbol{x})$ can be decomposed into a homogeneous part $w(\boldsymbol{x})$ which satisfies both boundary conditions and a homogeneous part $v(\boldsymbol{x})$ characterized by:

$$\nabla \cdot (\sigma(\boldsymbol{x})\nabla v(\boldsymbol{x})) = -\nabla \cdot (\sigma(\boldsymbol{x})\nabla w(\boldsymbol{x})), \boldsymbol{x} \in \Omega \quad (5)$$
$$v(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_D \quad (6)$$
$$\vec{n} \cdot \sigma(\boldsymbol{x})\nabla v(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Gamma_N. \quad (7)$$

The new formulation implies that one first performs a "lifting" operation of the epicardial boundary conditions onto the function space defined over the entire domain and then solves a homogeneous problem whose forcing function includes the heterogeneous term. There are three approximation issues involved in this process: (1) how accurately the epicardial boundary condition $u_0(\boldsymbol{x})$ is represented; (2) the choice of the lifting operator and the accuracy of its projection from the epicardial boundary to the volume; and (3) the accuracy of solving the homogeneous problem $v(\boldsymbol{x})$.

The FEM tessellates the torso volume Ω and constructs a set of basis functions, each of which associated with one vertex. The potential field $u(\boldsymbol{x})$ is represented by the linear combination of basis functions weighted by the potential value on corresponding vertices. Substituting this expansion into the differential equation (1) and applying the Galerkin method yields a linear system of the form:

$$\begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IT} \\ \mathbf{A}_{TI} & \mathbf{A}_{TT} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_T \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{IH} \\ 0 \end{pmatrix} \mathbf{u}_H \quad (8)$$

where $\mathbf{u}_H$, $\mathbf{u}_T$, and $\mathbf{u}_I$ denote the vector of potential values on the discretized heart surface (H), the torso surface (T), and inside the torso volume(I). The stiffness matrix A is partitioned into six submatrices according to the interaction among the three divisions.

Equation 8 describes the impacts of aforementioned three approximation issues. Once $\mathbf{u}_H$ determines the resolution of the epicardial potentials, the accuracy of the FEM approximation is dictated by the discretization of the heart/volume interaction (A_{IH}) as well as the volume conductor (the left-side matrix). We will address the resolution choice in combination with the concern of ill-posedness in the later section.

Equation 8 is then solved to derive the relationship between the epicardial potentials and the resulting body-surface potentials in the form of:

$$\mathbf{u}_T = \mathbf{K}\mathbf{u}_H \quad (9)$$

where $\mathbf{K} = \mathbf{M}^{-1}\mathbf{N}$ is termed as the transfer matrix. The inverse ECG problem attempts to recover $\mathbf{u}_H$ given $\mathbf{u}_T$; however, $\mathbf{K}$ is severely ill-conditioned.

2.2. Ill-Conditioning of the Numerical System

Since the primary challenge of solving the inverse ECG problem is to overcome its physical ill-posedness, we evaluate the quality of different FEM discretizations by assessing the ill-conditioning of their resultant transfer matrix $\mathbf{K}$. This is achieved by examining the singular values of K and utilizing the concept of valid and null spaces of K.

Assuming σ_i are the singular values of $\mathbf{K} \in \Re^{m\times n}(m > n)$, and u_i, v_i are the left and right eigenvectors of $\mathbf{K}$, then we have:

$$\mathbf{u}_T = \mathbf{K} \cdot \mathbf{u}_H = \sum_{i=1}^{n} u_i \sigma_i (v_i^T \mathbf{u}_H) = \sum_{i=1}^{n} \alpha_i u_i \quad (10)$$

where $\alpha_i = \sigma_i \cdot (v_i^T \mathbf{u}_H)$ is a scalar value representing the product of the σ_i and the projection of $\mathbf{u}_H$ on the i^{th} eigenvector of the discretized epicardial function space. As σ_i rapidly drops to zero, the epicardial space can be decomposed into a valid subspace spanned by low indexed eigenvectors and a null subspace spanned by high indexed eigenvectors. Only the fraction of $\mathbf{u}_H$ that falls in the valid space contributes to the observable $\mathbf{u}_T$ and can thereby be recovered.

Accordingly, a slowly-descending singular value spectrum with more non-trivial singular values indicates a better conditioning of a discrete inverse problem. The fraction of $\mathbf{u}_H$ in the valid subspace estimates the best solution recoverable, regardless of regularization methods, regularization parameters, error measurements, or input noise.

2.3. Regularization Methods and Parameter Selection

Since $\mathbf{K}$ is severely ill-conditioned and the measured data $\mathbf{u}_T$ is inevitably contaminated with noise, solving 9 requires regularization. Regularization introduces extra constraints so as to yield a well-posed problem, the solution to which is stable and not too far from the desired one.

We solve the inverse problem by adopting the classic Tikhonov regularization expressed as

$$\mathbf{u}_H(\lambda) = argmin\{\|\mathbf{K}\mathbf{u}_H - \mathbf{u}_T\|_2^2 + \lambda^2\|\mathbf{L}\mathbf{u}_H\|^2\} \quad (11)$$

where $\mathbf{L}$ is the discrete operator of the second order derivative, and λ is a regularization parameter that controls the weight placed on the regularization relative to that placed on solving the original problem. The value of λ is determined by an exhaustive search.

The "optimal" value of λ (however it is selected) reflects the ill-condition severity of the inverse problem being solved. A small λ means that the problem is not severely ill-conditioned and therefore does not require much regularization, whereas a large λ offers the opposite implication. In the extreme case of solving a well-conditioned problem, the solution is obtained by minimizing the residual error only and no regularization is needed ($\lambda = 0$). The parameter λ provides another means to assess the ill-conditioning during the regularization process, in addition to evaluating singular values.

2.4. Ill-Posedness Considerations

Since the human body is known to respond differently to electric signals of different spatial frequencies, Fourier analysis aids us in quantifying how ill-conditioned the system is, and thus helps provide a means of forming guidelines for discretization. We discover that the ill-posedness is an exponential function of the spatial frequency (*w.r.t.*the azimuthal variable in 2D or any closed surfaces in 3D). Reconstructing epicardial potentials in higher fidelity therefore implies worse conditioning of the discrete inverse problem. This discretization concern differs from that in the forward problem.

Moreover, the heart-surface resolution gives the band-limit of the epicardial potentials one seeks to recover, whereas the volume discretization determines the band-limit of the actually solvable potential. Practitioners should ensure the former not exceeding the latter, so as to avoid extra ill-conditioning not due to the physical nature but to inadequate discretization.

3. Results of Different Mesh Refinements

3.1. Uniform Refinement in 2D

Although uniform refinements effectively reduce errors in the forward problem solution (Figure 1 (top)), they worsen the singular values and hence the ill-conditioning of the transfer matrix **K** in the inverse problem (Figure 1 (bottom)). This example shows discretization strategies for forward problems should be cautiously carried over to inverse problems.

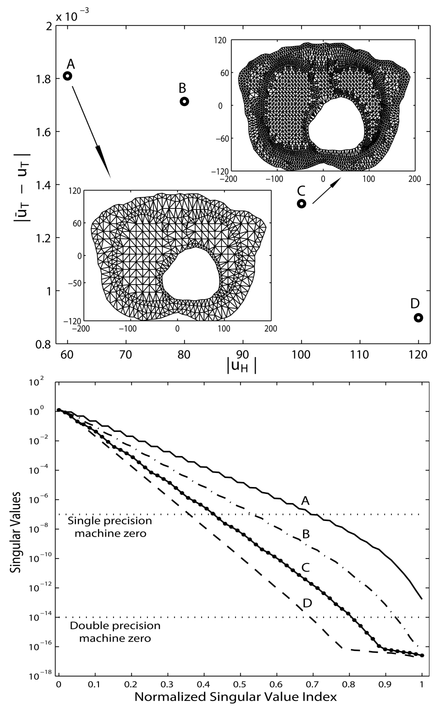

Fig. 1. (top): Forward solution error convergence with increasingly refined meshes labeled as A-D. Only Mesh A and C are displayed. $|u_H|$ represents the epicardial resolution, $|\bar{u}_T - u_T|$ means the solution error on the torso surface. (Bottom): Singular value spectra of **K** resulting from meshes A-D, normalized for visual comparison.

3.2. Volume Refinement in 2D

Figure 2 (A–C) displays three discretization levels of the torso, with both the heart and torso boundary resolutions being fixed. Such volume refinement improves the singular value spectrum of K (Panel D), leading to more accurate reconstruction of epicardial potentials (Panel E). This observation is also supported by Table 1. In the noise-free case, volume refinement reduces the regularization amount λ from 0.0077 to 0.0005, indicating the improvement of the conditioning of K; the relative error (RE) in the inverse solution is correspondingly reduced from 8.81% to 4.19%. The improvement is also observed in the case of 30dB noise. Note that the proportion of non-trivial singular values in the overall eigenspace of **K** is determined by the resolution of the polygon that encloses the interior boundary with the least number of nodes. The gap between the singular value spectrum of A and that of C indicates the ill-conditioning caused by insufficient discretization but not associated with the ill-posed nature of the continuum problem, as discussed in section 2.4.

3.3. Volume Refinement and Hybrid Meshing in 3D

Given that the epicardial surface resolution determines the ill-conditioning while the normal direction resolution captures the high gradient field around the heart, it is natural to decouple both resolutions by placing prism elements which do not have as restrictive aspect-ratio issues as typical in tetrahedral elements (or substituting quadrilaterals for triangles in 2D), as illustrated in Figure 3(top). Figure 3(bottom) compares such hybrid meshes with ordinary tetrahedral meshes in terms of their resultant transfer matrices' singular values. Under similar discretization levels, the hybrid mesh yields better conditioned transfer matrices. In both types of meshes, volume refinements preserving boundary resolutions extend the valid singular value spectrum, which is consistent with the 2D situation discussed in Section 3.2.

4. Conclusions

This study investigates how the FEM discretization of the inverse ECG problem influences the numerical conditioning of the resulting discrete system. We summarize the refinement guidelines for inverse problems as follows:

First, refining the heart surface increases the ill-conditioning the discretized inverse system. One should realistically assess the heart-surface resolution sufficient for the specific problem of interest but be cautious to refine beyond the minimum resolution needed.

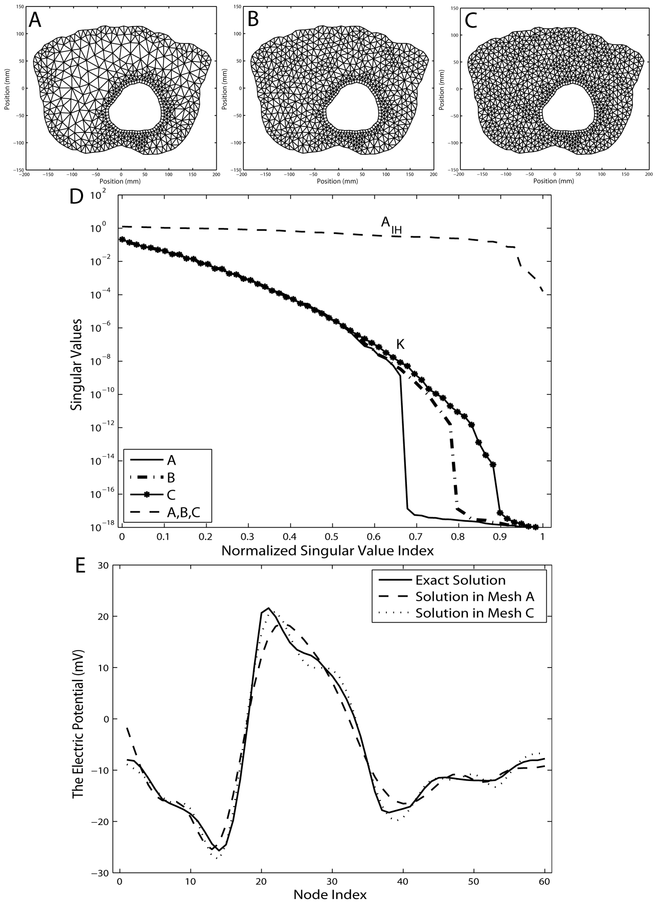

Fig. 2. (A)-(C): torso meshes in ascending volume resolutions and with the same boundary resolutions. (D): singular values of $\mathbf{K}$ and $\mathbf{A}_{IH}$. (E): reconstructed epicardial potentials.

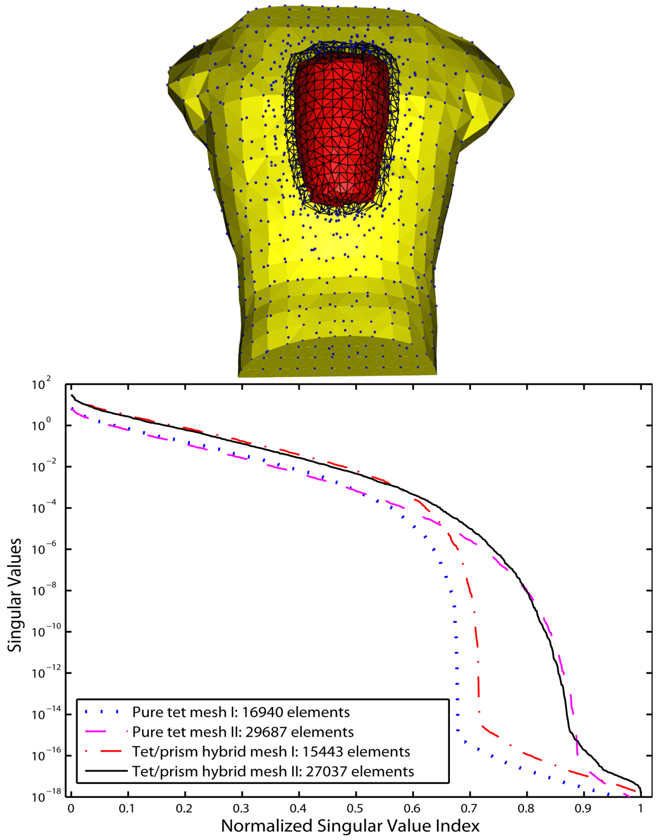

Fig. 3. (Top): a hybrid torso mesh example with one layer of prisms around the heart approximated by a cylinder. The blue dots represent vertices of the tetrahedral volume mesh. (Bottom):singular values of $\mathbf{K}$ resulted from two types of mesh. Each mesh type has two meshes in the same boundary surface discretizations but different discretization levels for the volume.

Table 1. Relative errors (RE) of the inverse solutions and the corresponding optimal regularization parameter λ, resulting from the simulations on torso meshes shown in Figure 2. RE measures the ratio of the error to the exact solution in the L_2 norm.

Input Noise	Noise Free		30dB		20dB	
	RE	λ	RE	λ	RE	λ
Mesh A	8.81%	0.0077	9.02%	0.0080	14.33%	0.0205
Mesh B	5.85%	0.0016	7.90%	0.0042	15.45%	0.0207
Mesh C	4.19%	0.0005	7.17%	0.0035	15.26%	0.0215

Note: the random input noise is in normal distribution. Each data presented above is the arithmetic average of 50 repeated simulations.

Second, refining the body surface improves the inverse system to some extent, but only when the body-surface ECG measurements is also increased.

With the above two items in place, the volume conductor should be refined sufficiently to capture both the features of the torso data and the features implied by the discretization of the heart surface. For computational efficiency, beyond that level is unnecessary.

Last, increasing the resolution normal to the heart surface improves the approximation of the boundary-to-volume lifting operator. This requires decoupling tangential and normal resolutions, for which we advocate hybrid discretization – quadrilateral elements in 2D and prism elements in 3D. These geometries also connect well with tetrahedral or hexahedral elements filling the volume.

Future work includes properly defining the patterns and spatial frequencies of epicardial potentials on a 3D surface, and understanding the impact of anisotropic conductivity. Another direction is optimizing resource distributions given limited computational resources.

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