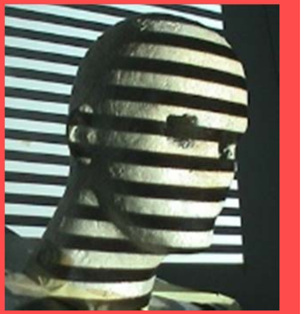


Geometric Camera Calibration
Chapter 2

Guido Gerig

CS 6320 Spring 2012

Slides modified from Marc Pollefeys, UNC Chapel Hill, Comp256,
Other slides and illustrations from J. Ponce, addendum to course book,
and Trevor Darrell, Berkeley, C280 Computer Vision Course.

Equation: World coordinates to image pixels

pixel coordinates

world coordinates

$$\vec{p} = \frac{1}{z} M \; {}^{W}\vec{p}$$

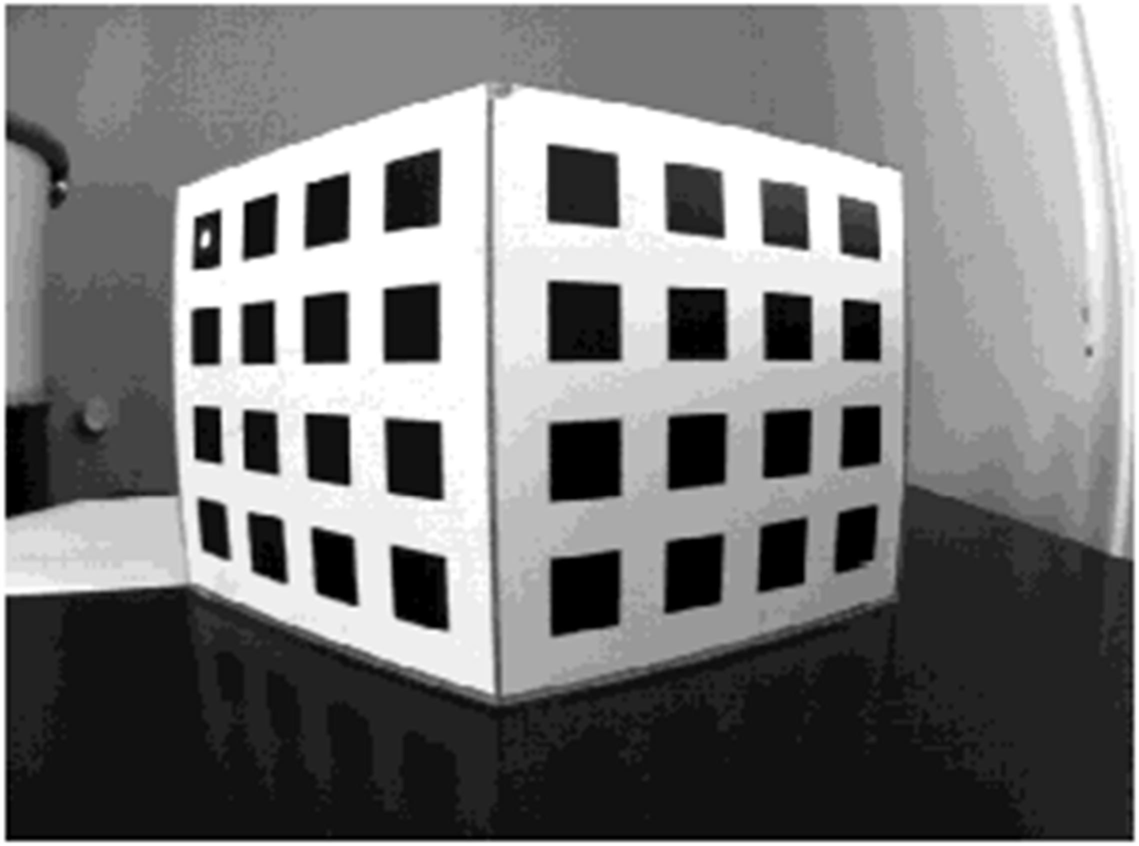
$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{1}{z} \begin{pmatrix} . & m_1^T & . & . \\ . & m_2^T & . & . \\ . & m_3^T & . & . \end{pmatrix} \begin{pmatrix} {}^{W}p_x \\ {}^{W}p_y \\ {}^{W}p_z \\ 1 \end{pmatrix}$$

$$\left\{ \begin{array}{l} u = \dfrac{m_1 \cdot \vec{P}}{m_3 \cdot \vec{P}} \\[2ex] v = \dfrac{m_2 \cdot \vec{P}}{m_3 \cdot \vec{P}} \end{array} \right.$$

Conversion back from homogeneous coordinates leads to:

W. Freeman

Calibration target



The Opti-CAL Calibration Target Image

Find the position, u_i and v_i, in pixels,
of each calibration object feature point.

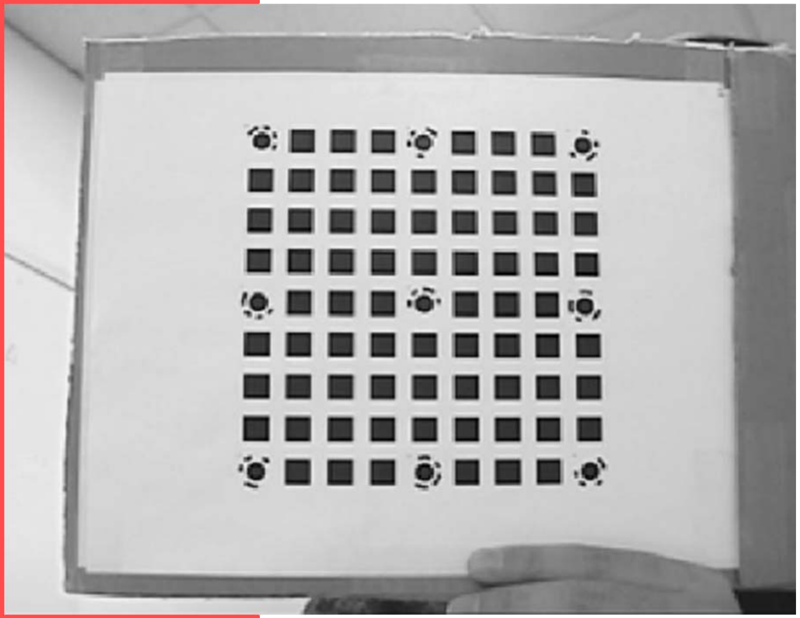
http://www.kinetic.bc.ca/CompVision/opti-CAL.html

Camera calibration

From before, we had these equations relating image positions, u,v, to points at 3-d positions P (in homogeneous coordinates):

$$u = \frac{m_1 \cdot \vec{P}}{m_3 \cdot \vec{P}}$$

$$v = \frac{m_2 \cdot \vec{P}}{m_3 \cdot \vec{P}}$$



So for each feature point, i, we have:

$$(m_1 - u_i m_3) \cdot \vec{P}_i = 0$$

$$(m_2 - v_i m_3) \cdot \vec{P}_i = 0$$

W. Freeman

Camera calibration

Stack all these measurements of i=1…n points

$$(m_1 - u_i m_3) \cdot \vec{P}_i = 0$$
$$(m_2 - v_i m_3) \cdot \vec{P}_i = 0$$

into a big matrix:

$$\begin{pmatrix} P_1^T & 0^T & -u_1 P_1^T \\ 0^T & P_1^T & -v_1 P_1^T \\ \cdots & \cdots & \cdots \\ P_n^T & 0^T & -u_n P_n^T \\ 0^T & P_n^T & -v_n P_n^T \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

W. Freeman

Camera calibration

In vector form:

$$\begin{pmatrix} P_1^T & 0^T & -u_1 P_1^T \\ 0^T & P_1^T & -v_1 P_1^T \\ \cdots & \cdots & \cdots \\ P_n^T & 0^T & -u_n P_n^T \\ 0^T & P_n^T & -v_n P_n^T \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Showing all the elements:

$$\begin{pmatrix} P_{1x} & P_{1y} & P_{1z} & 1 & 0 & 0 & 0 & 0 & -u_1 P_{1x} & -u_1 P_{1y} & -u_1 P_{1z} & -u_1 \\ 0 & 0 & 0 & 0 & P_{1x} & P_{1y} & P_{1z} & 1 & -v_1 P_{1x} & -v_1 P_{1y} & -v_1 P_{1z} & -v_1 \\ & & & & & & & \cdots & \cdots & \cdots & & \\ P_{nx} & P_{ny} & P_{nz} & 1 & 0 & 0 & 0 & 0 & -u_n P_{nx} & -u_n P_{ny} & -u_n P_{nz} & -u_n \\ 0 & 0 & 0 & 0 & P_{nx} & P_{ny} & P_{nz} & 1 & -v_n P_{nx} & -v_n P_{ny} & -v_n P_{nz} & -v_n \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

W. Freeman

Camera calibration

$$\begin{pmatrix} P_{1x} & P_{1y} & P_{1z} & 1 & 0 & 0 & 0 & 0 & -u_1P_{1x} & -u_1P_{1y} & -u_1P_{1z} & -u_1 \\ 0 & 0 & 0 & 0 & P_{1x} & P_{1y} & P_{1z} & 1 & -v_1P_{1x} & -v_1P_{1y} & -v_1P_{1z} & -v_1 \\ & & & & & & & \cdots & \cdots & \cdots & & \\ P_{nx} & P_{ny} & P_{nz} & 1 & 0 & 0 & 0 & 0 & -u_nP_{nx} & -u_nP_{ny} & -u_nP_{nz} & -u_n \\ 0 & 0 & 0 & 0 & P_{nx} & P_{ny} & P_{nz} & 1 & -v_nP_{nx} & -v_nP_{ny} & -v_nP_{nz} & -v_n \end{pmatrix} \begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Q m = 0

We want to solve for the unit vector m (the stacked one) that minimizes $|Qm|^2$

The eigenvector assoc. to the minimum eigenvalue of the matrix Q^TQ gives us that because it is the unit vector x that minimizes $x^T Q^TQ\ x$.

W. Freeman

Homogeneous Linear Systems

$$A \, x = 0$$

Square system:

- unique solution: 0
- unless Det(A)=0

$$A \, x = 0$$

Rectangular system ??

- 0 is always a solution

Minimize $|Ax|^2$
under the constraint $|x|^2=1$

How do you solve overconstrained homogeneous linear equations ??

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.
- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.
- Any vector can be written as

$$\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_q\boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x}) - E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x} - \boldsymbol{e}_1^T(\mathcal{U}^T\mathcal{U})\boldsymbol{e}_1 \\ &= \lambda_1^2\mu_1^2 + \ldots + \lambda_q^2\mu_q^2 - \lambda_1^2 \\ &\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0 \end{aligned}$$

The solution is e_1.

<u>remember</u>: EIG(U^TU)=SVD(U), i.e. solution is V_n

Degenerate Point Configurations

Are there other solutions besides *M* ??

$$\mathbf{0} = \mathcal{P}\boldsymbol{l} = \begin{pmatrix} \boldsymbol{P}_1^T & \mathbf{0}^T & -u_1\boldsymbol{P}_1^T \\ \mathbf{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \mathbf{0}^T & -u_n\boldsymbol{P}_n^T \\ \mathbf{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_1^T\boldsymbol{\lambda} - u_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \boldsymbol{P}_1^T\boldsymbol{\mu} - v_1\boldsymbol{P}_1^T\boldsymbol{\nu} \\ \dots \\ \boldsymbol{P}_n^T\boldsymbol{\lambda} - u_n\boldsymbol{P}_n^T\boldsymbol{\nu} \\ \boldsymbol{P}_n^T\boldsymbol{\mu} - v_n\boldsymbol{P}_n^T\boldsymbol{\nu} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{cases} \boldsymbol{P}_i^T\boldsymbol{\lambda} - \dfrac{\boldsymbol{m}_1^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \\ \boldsymbol{P}_i^T\boldsymbol{\mu} - \dfrac{\boldsymbol{m}_2^T\boldsymbol{P}_i}{\boldsymbol{m}_3^T\boldsymbol{P}_i}\boldsymbol{P}_i^T\boldsymbol{\nu} = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{P}_i^T(\boldsymbol{\lambda}\boldsymbol{m}_3^T - \boldsymbol{m}_1\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \\ \boldsymbol{P}_i^T(\boldsymbol{\mu}\boldsymbol{m}_3^T - \boldsymbol{m}_2\boldsymbol{\nu}^T)\boldsymbol{P}_i = 0 \end{cases}$$

- Coplanar points: $(\lambda,\mu,\nu) = (\Pi,0,0)$ or $(0,\Pi,0)$ or $(0,0,\Pi)$
- Points lying on the intersection curve of two quadric surfaces = straight line + twisted cubic

Does not happen for 6 or more random points!

Camera calibration

Once you have the M matrix, can recover the intrinsic and extrinsic parameters.

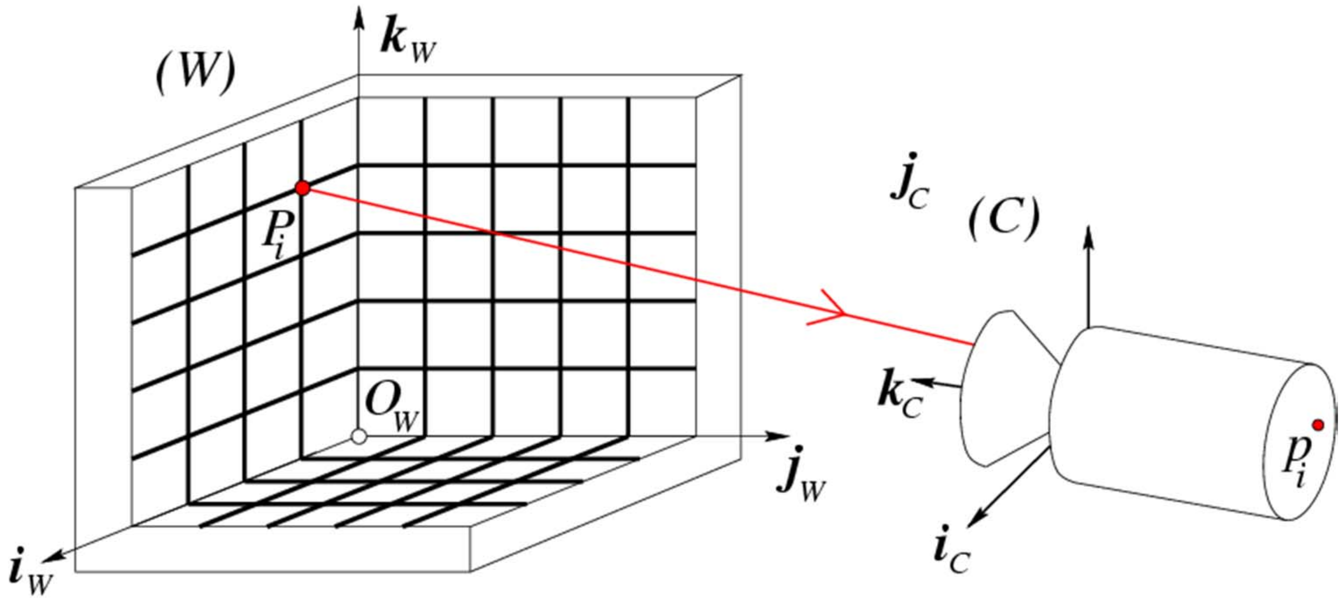
Estimation of the Intrinsic and Extrinsic Parameters, see pdf slides S.M. Abdallah.

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

W. Freeman

Other Slides following Forsyth&Ponce

Calibration Problem



Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

Analytical Photogrammetry

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

Find $\boldsymbol{i}$ and $\boldsymbol{e}$ such that

$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

Non-Linear Least-Squares Methods

- Newton
- Gauss-Newton
- Levenberg-Marquardt

Iterative, quadratically convergent in favorable situations

Linear Systems

$$A \; x = b$$

Square system:

- unique solution
- Gaussian elimination

$$A \; x = b$$

Rectangular system ??

- underconstrained:
  infinity of solutions
- overconstrained:
  no solution

Minimize $|Ax-b|^2$

How do you solve overconstrained linear equations ??

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\boldsymbol{e} \;=\; A\boldsymbol{x} - \boldsymbol{b} = \left[\begin{array}{c|c|c|c} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \dots & \boldsymbol{c}_n \end{array}\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b}$$

$$= \; x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b}$$

- At a minimum,

$$\frac{\partial E}{\partial x_i} \;=\; \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e}$$

$$= \; 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e}$$

$$= \; 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

where $\boxed{\boldsymbol{x} = A^\dagger\boldsymbol{b} \text{ and } A^\dagger = (A^TA)^{-1}A^T \text{ is the } \textit{pseudoinverse} \text{ of } A\ !}$

Homogeneous Linear Systems

$$A \; x = 0$$

Square system:

- unique solution: 0
- unless Det(A)=0

$$A \; x = 0$$

Rectangular system ??

- 0 is always a solution

Minimize $|Ax|^2$
under the constraint $|x|^2=1$

How do you solve overconstrained homogeneous linear equations ??

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.
- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.
- Any vector can be written as

$$\boldsymbol{x} = \mu_1 \boldsymbol{e}_1 + \ldots + \mu_q \boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x}) - E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x} - \boldsymbol{e}_1^T(\mathcal{U}^T\mathcal{U})\boldsymbol{e}_1 \\ &= \lambda_1^2\mu_1^2 + \ldots + \lambda_q^2\mu_q^2 - \lambda_1^2 \\ &\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0 \end{aligned}$$

The solution is e_1 .

<u>remember</u>: EIG(U^TU)=SVD(U), i.e. solution is V_n

Linear Camera Calibration

Given n points $P_1, \ldots, P_n$ with *known* positions and their images $p_1, \ldots, p_n$

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\ \ldots & \ldots & \ldots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

Once *M* is known, you still got to recover the intrinsic and extrinsic parameters !!!

This is a decomposition problem, not an estimation problem.

$$\boxed{\rho}\ \mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

- Intrinsic parameters
- Extrinsic parameters

Section 3.2.2

- Estimation of the Intrinsic and Extrinsic Parameters (S.M. Abdallah)

Useful Links

Demo calibration (some links broken):

- http://mitpress.mit.edu/e-journals/Videre/001/articles/Zhang/CalibEnv/CalibEnv.html

Bouget camera calibration SW:

- http://www.vision.caltech.edu/bouguetj/calib_doc/

CVonline: Monocular Camera calibration:

- http://homepages.inf.ed.ac.uk/cgi/rbf/CVONLINE/entries.pl?TAG250