A.N.M. Imroz Choudhury
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We wish to compute definite integrals over rectangular regions of the function

$$f(x,y) = \frac{1}{4}\left(\frac{x}{n_x}\left(1+\cos(k_x x^2)\right)\right)\left(\frac{y}{n_y}\left(1+\cos(k_y y^2)\right)\right) \tag{1}$$

in which n_x, n_y, k_x, and k_y are constants, and $(x,y) \in [0,n_x] \times [0,n_y]$. First we perform some rearrangement:

$$\begin{aligned} f(x,y) &= \frac{1}{4}\left(\frac{x}{n_x}\left(1+\cos(k_x x^2)\right)\right)\left(\frac{y}{n_y}\left(1+\cos(k_y y^2)\right)\right) \\ &= \frac{1}{4n_x n_y}\left(x + x\cos k_x x^2\right)\left(y + y\cos k_y y^2\right) \\ &= cF_x(x)F_y(y) \end{aligned}$$

In other words, $f(x,y)$ is separable in x and y, which will simplify the integration. We want the integral of f over a single pixel area, and so:

$$\begin{aligned} \int_j^{j+1}\int_i^{i+1} f(x,y)\,dx\,dy &= \int_j^{j+1}\int_i^{i+1} cF_x(x)F_y(y)\,dx\,dy \\ &= c\int_j^{j+1}\int_i^{i+1} F_x(x)F_y(y)\,dx\,dy \\ &= c\int_j^{j+1} F_y(y)\left(\int_i^{i+1} F_x(x)\,dx\right)dy \\ &= c\left(\int_i^{i+1} F_x(x)\,dx\right)\left(\int_j^{j+1} F_y(y)\,dy\right) \qquad (2) \end{aligned}$$

Taking the dx integral above, we get:

$$\begin{aligned} \int_i^{i+1} F_x(x)\,dx &= \int_i^{i+1}\left(x + x\cos k_x x^2\right)dx \\ &= \int_i^{i+1} x\,dx + \int_i^{i+1} x\cos k_x x^2\,dx \\ &= \left.\frac{x^2}{2}\right|_i^{i+1} + \left[\frac{-\sin(k_x x^2)}{2k_x}\right]_i^{i+1} \\ &= \left(\frac{(i+1)^2 - i^2}{2}\right) + \left(\frac{-\sin(k_x(i+1)^2) + \sin(k_x i^2)}{2k_x}\right) \end{aligned}$$

Replacing i with j and k_x with k_y in this final expression gives the value of the dy integral, and by substituting into (2), these expressions give the integral of (1).

It is possible to simplify the resulting expression so that direct translation to C or C++ code is more efficient, but this is far afield of the mathematical timbre this short report seeks after.