Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 1 Solutions

Simple valuations

Due: Friday, Aug 30, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) (Loan valuation) Suppose the (annual) interest rate for a loan is currently 3% for a loan term of 3 years, and the interest will compound monthly. You estimate that you will be able to afford a maximum monthly payment of \$300 over the loan term. What is the maximum loan amount you can take out and still afford the monthly payments?

Solution: We model this loan as an annuity with the following parameters:

- Interest rate: $r = 0.03$ per year
- Compounded monthly: $k = 12$ terms per year
- $n = 3\text{years} \times 12 = 36$ periods (months)
- $P = 300$ payment every period

The unknown is the loan principal today (the present value); hence our goal is to compute the present value of this annuity. We recall that an n-period annuity with per-period payment of P with a per-period interest rate of r/k has present value:

$$PV = \sum_{j=1}^{36} \frac{P}{(1+r/k)^n} = \frac{Pk}{r}\left(1 - \frac{1}{(1+\frac{r}{k})^n}\right).$$

Using the parameters above in this formula yields a present value of,

$$PV \approx \$10,315.94,$$

which is the maximum affordable loan principal.

2.) (Compound interest) You decide that starting today you will deposit \$300 per month in a money market account with an annual interest rate of 5% compounded daily (ordinary interest). Deposits are made in lump sums of \$300 on the first day of each month (30-day period), and today is the first day of the month. After 3 years time (36 deposits total), how much money will you have in the account? What is the minimum monthly deposit required so that the account will be worth \$15,000 after 3 years? **Solution:** Here are some particulars about this problem:

- The (annual) interest rate is $r = 0.05$
- With ordinary interest, there are 30 days per month, and 360 days in a year.
- The per-day interest rate is therefore $\frac{0.05}{360}$.
- For interest accrual that occurs daily, there are 3 years, or $3 \times 360 = 1080$ interest periods.

The amount of money in the account accrues according to,

- \$300 deposited at the beginning of the first month accrues 1080 days worth of interest, i.e., at the end of 3 years is worth $300\left(1+\frac{0.05}{360}\right)^{1080}$.
- \$300 deposited at the beginning of the second month accrues 1080 - 30 = 1050 days worth of interest, i.e., at the end of 3 years is worth $300\left(1+\frac{0.05}{360}\right)^{1050}$.

This process continues for 36 periods. Therefore, the total value of the account at the end of 3 years is the sum of all these values:

$$\begin{aligned}
\text{Value} = 300\left(1+\frac{0.05}{360}\right)^{1080} + 300\left(1+\frac{0.05}{360}\right)^{1050} + \cdots \\
&= \sum_{j=0}^{35} 300\left(1+\frac{0.05}{360}\right)^{1080-30j} \\
&= 300\left(1+\frac{0.05}{360}\right)^{1080} \sum_{j=0}^{35}\left[\frac{1}{\left(1+\frac{0.05}{360}\right)^{30}}\right]^{j} \\
&= 300\rho^{36}\sum_{j=0}^{35}\left(\frac{1}{\rho}\right)^{j} \qquad \rho := \left(1+\frac{0.05}{360}\right)^{30} \\
&= 300\rho^{36}\frac{1-\frac{1}{\rho^{36}}}{1-\frac{1}{\rho}} \\
&= 300\rho\frac{\rho^{36}-1}{\rho-1}.
\end{aligned}$$

With this computation, we have that,

$$\text{Value} = \$11,676.29$$

In order to determine the monthly deposit required to achieve \$15,000 value, we can actually use the previous computation. In particular, if we make a monthly payment of P dollars instead of \$300, we conclude,

$$\text{Value} = P\rho\frac{\rho^{36}-1}{\rho-1},$$

where ρ as defined above doesn't depend on P. Therefore, to achieve a value of \$15,000, we solve for P to obtain:

$$P = \$385.40.$$

3.) (Simple options) A *call option* is a contract between a buyer and seller that entitles the buyer the right (but not the requirement) to purchase a fixed amount of stock shares at a specified, "strike" price: The buyer agrees to pay the seller a premium today, which gives the buyer the right to force the seller to sell the shares at the strike price anytime between today and the expiry term of the option. (Hence, the sale completes based on today's agreed upon strike price, not based on market price on the day of the sale.) A "European" option is one where the buyer is permitted to exercise the option only at expiry (not before). Assume rational acting: the buyer exercises only if it is a rational choice (i.e,. only if the market price at expiry is greater than the strike price).

(a) In entering to a European call option contract, briefly explain what the buyer hopes will happen to the future stock price (and why), and what the seller hopes will happen (and why).

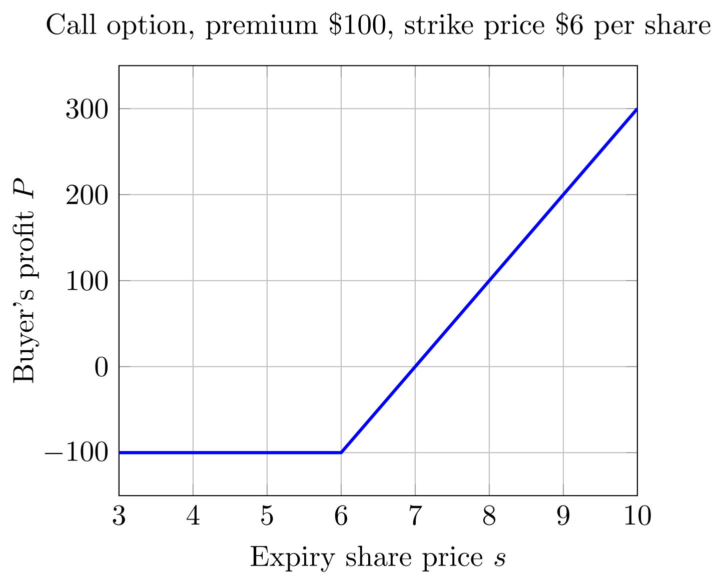
(b) Let ABC be a stock selling on the market today for \$5 per share. Suppose a European call option premium is \$100 for 100 shares of ABC at strike price \$6 per share.[1] Plot the buyer's profit as a function of ABC's expiry share price.

(c) Consider a *put option*, in which the seller pays a premium to gain the right (but not the requirement) to sell stock to the buyer at the strike price. With the same fees, parameters, and rational acting as in the previous part, plot the seller's profit on a European put option as a function of ABC's expiry share price.

Solution:

(a) The buyer hopes that the stock price will rise so that they can buy at the (lower) strike price in the future and immediately sell back to the market for a profit that overcomes the loss in paying the premium. The seller hopes that the stock price will not rise (or fall) so that either the buyer will not exercise the option and the seller can absorb the premium as a profit, or that the stock price rises too little to overcome the profit the seller has gained in the premium.

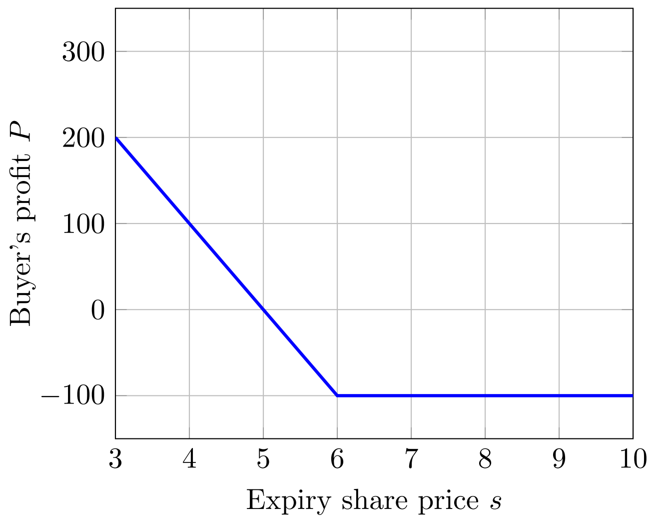
(b) The buyer's profit $P(s)$ is plotted below as a function of the expiry share price s in dollars.



(c) In this case the seller pays a premium of \$10 for forcing a sale at the strike price of \$60, i.e., they hope that the stock price falls below \$60 so that they can sell to the buyer at \$60 and buy back a share from the market at a lower rate, making a profit. The profit $P(s)$ as a function of expiry share price is shown below.

[1] Options are commonly sold in units of 100 shares.

Put option, premium \$100, strike price \$6 per share



4.) (Arbitrage) Consider the options setup above (same parameters and dollar amounts for both call and put options). Suppose you have the ability to invest (take long positions) in 3 securities: European call options for ABC, European put options for ABC, and a money market account with an annual rate of 4% compounded daily (ordinary interest). All options have an expiry of 1 year. Assume you have \$10,000 available to invest, and that you will invest all of it in a combination of these 3 securities. You may only purchase whole units of options, no fractional purchases are allowed; the money market investment has no limitations on the investment amount.

(a) Identify all actions that result in arbitrage at expiry of the options.

(b) From the collection of actions identified above, describe what knowledge about ABC would make one action more favorable than another.

Solution:

(a) Arbitrage in this context is achieving a value of more than \$10,000 at expiry. The set of all actions is determined by the number of put options P and call options C purchased today. In particular, these actions are,

$$\left\{(P, C) \in \mathbb{N}_0^2 \;\middle|\; 100P + 100C \leq 1000\right\}.$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. I.e., this is set of non-negative integers P, C such that $P{+}C \leq 100$. The inequality constraint arises because we cannot affordably purchase more than 100 options with \$10,000 capital. The question is then which sets of (P, C) guarantee that we have value surpassing $\$10,000$ at expiry. Note that the worst outcome for purchasing options is that we don't exercise any of them. (Any other outcome results in some payoff at expiry from the options.) Therefore, regardless of underlier price of the options, the worst outcome is that we simply pay the premium invested today. Therefore, the value of our portfolio at expiry is,

$$\text{Value} = \underbrace{-100(P+C)}_{\text{Premium paid today}} + \underbrace{(10000 - 100(P+C))}_{\text{Leftover cash today}} \underbrace{\left(1 + \frac{0.04}{360}\right)^{360}}_{\text{Ordinary compounded interest}} .$$

The first thing to notice from this formula is that its value depends *only* on $(P+C)$, and not P and C individually. Hence, the only thing that matters is how many options we purchase. In particular, we want to determine values of $(P+C)$ that result in a Value greater than \$10,000. From the above formula, we can directly compute:

- $P+C=0 \implies$ Value $= \$10,408.08$
- $P+C=1 \implies$ Value $= \$10,307.04$
- $P+C=2 \implies$ Value $= \$10,206.00$
- $P+C=3 \implies$ Value $= \$10,104.96$
- $P+C=4 \implies$ Value $= \$10,003.92$
- $P+C=5 \implies$ Value $= \$9,902.88$
- $\vdots$

So that $P+C \leq 4$ results in arbitrage, but $P+C > 4$ does not. Therefore, the actions corresponding to arbitrage are:

- $P+C=0 \implies (P,C)=(0,0)$
- $P+C=1 \implies (P,C)=(1,0),\ (0,1)$
- $P+C=2 \implies (P,C)=(2,0),\ (1,1),\ (0,2)$
- $P+C=3 \implies (P,C)=(3,0),\ (2,1),\ (1,2),\ (0,3)$
- $P+C=4 \implies (P,C)=(4,0),\ (3,1),\ (2,2),\ (1,3),\ (0,4)$

(b) The final value is the price of the underlier at expiry would affect the value: it determines whether we exercise options or not. In particular:

- If the price at expiry is larger than the strike price, we will exercise call options.
- If the price at expiry is small than the strike price, we will exercise put options.

Therefore, having any knowledge/predictions about the price of ABC at expiry would inform actions. E.g., if we knew with great certainty that ABC will substantially exceed the strike price in the future, then we would probably invest mostly in call options (large C) and not invest in put options (zero P).