Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 2 solutions

More valuations

Due: Friday, Sept 6, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) (Bond valuation) Bonds are priced based on today's present value of the instrument; the "face value" of a bond is the amount paid to the bearer at expiry of the bond's term ("maturity"). In the simplest setting, the price is based on the sum of two things: (i) the present value of the face value of the bond (which is received at maturity), (ii) the present value of an annuity ("coupon payment") paid to the holder at regular intervals; the payment per period equals the face value times the "coupon/interest rate" (prorated from a quoted annual rate). The "yield to maturity" is an assumed (annual) interest rate on which the present value is discounted from future value.

(a) Determine the price of the following bond: A bond with a face value of \$1000 with a maturity term of 2 years and a coupon rate of 4%. The coupon payments are made semiannually (i.e., twice a year at \$20 per payment) until the bond matures. Throughout, assume a discount rate ("yield to maturity") of 3%.

(b) "Par" refers to a bond face value. Is the bond in part (a) priced below, at, or above par?

(c) (**6890 students only**) Prove in general that if the yield to maturity equals the coupon rate, then the present value of the bond is exactly par.

<u>**Solution:**</u>

(a) We compute the sum of the present value of the bond face value, plus the present value of the 4 coupon payments made over 2 years:

$$
\begin{aligned}
PV &= \frac{1000}{\left(1 + \frac{0.03}{2}\right)^4} + \sum_{j=1}^{4} \frac{20}{\left(1 + \frac{0.03}{2}\right)^j} \\
&\approx 942.18 + 77.09 = \$1019.27,
\end{aligned}
$$

which corresponds to the bond price.

(b) Since the bond price computed above is larger than the face value (\$1000), then this bond is priced above par.

(c) Assume a bond with face value F, an annual coupon rate r, with k coupon payments per year. As stated, we assume the annual discount rate (yield to maturity) is also r. Let the bond term be n periods (i.e., n/k years). The per-period coupon payment is $F\frac{r}{k}$. We seek to show that the present value of this bond is exactly the face value

F. The present value of such a bond is given by,

$$\begin{aligned}
PV &= \frac{F}{\left(1+\frac{r}{k}\right)^n} + \sum_{j=1}^{n} \frac{F\frac{r}{k}}{\left(1+\frac{r}{k}\right)^j} \\
&\overset{q=r/k}{=} \frac{F}{(1+q)^n} + \sum_{j=1}^{n} \frac{Fq}{(1+q)^j} \\
&= \frac{F}{(1+q)^n} + Fq\sum_{j=1}^{n} \frac{1}{(1+q)^j} \\
&= \frac{F}{(1+q)^n} + Fq\left[\frac{1-\frac{1}{(1+q)^{n+1}}}{1-\frac{1}{1+q}} - 1\right] \\
&= \frac{F}{(1+q)^n} + Fq\left[\frac{1+q-\frac{1}{(1+q)^n}}{1+q-1} - \frac{q}{q}\right] \\
&= \frac{F}{(1+q)^n} + F\left[1 - \frac{1}{(1+q)^n}\right] = F,
\end{aligned}$$

which proves the result.

2.) (Forward price) Consider entering a into forward contract with a maturity term of 1 year as a buyer for a single share of a company XYZ. Suppose that today's share price is \$100 per share, and that the company has committed to paying \$1, \$2, \$3, and \$5 in dividends at the end of quarters 1, 2, 3, and 4, respectively (a quarter is three months). Suppose in this market there is also a risk-free security with an annual interest rate of 4% (e.g., a savings account rate), compounded continuously. Assume you are a *rational pricer*, meaning that you value securities based on an assumption that there is no arbitrage in the market. Determine a fair forward price for this contract. (Recall that dividends implicitly decrease the value of a corporation.)

Solution: The fair forward price should be the future value of one share of XYZ. ("Future" means at maturity, i.e., 1 year from now.) Under the rational pricing model, we assume that the value of money grows according to the specified arbitrage-free 4% annual rate, compounded continuously. Therefore, the value of one share at maturity is given by today's price inflated by a continuously compounded 4% rate for 1 year:

$$\text{Future value of 1 share of XYZ} = \$100 \times e^{0.04\times 1} \approx \$104.08.$$

To account for the dividends, we receive payments at months 3, 6, 9, and 12, which we can determine the 1-year value of:

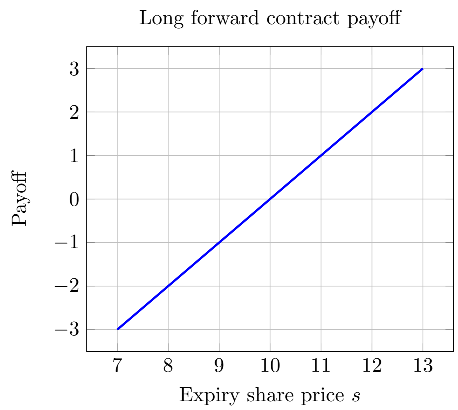
$$\$1 \times e^{0.04\times\frac{9}{12}} + \$2 \times e^{0.04\times\frac{6}{12}} + \$3 \times e^{0.04\times\frac{3}{12}} + \$5 \times e^{0.04\times\frac{0}{12}} \approx \$11.10$$

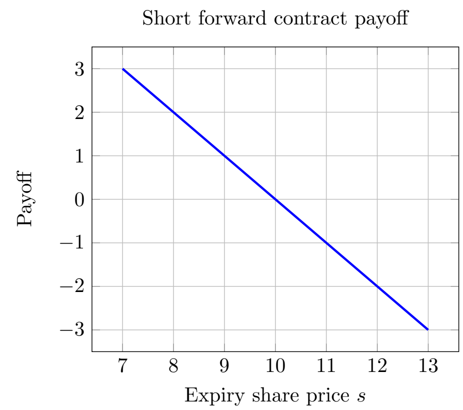
Then a fair forward price, i.e., the value of one share of XYZ at maturity under this rational pricing assumption is the future value of today's share price *minus* the future value of the dividends:

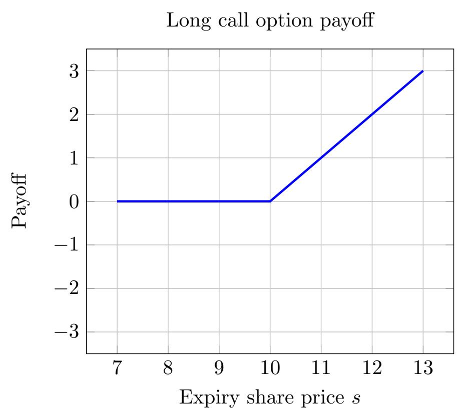
$$\begin{aligned}
\text{Fair forward price} &= \$100 \times e^{0.04\times 1} \\
&\quad - \left(\$1 \times e^{0.04\times\frac{9}{12}} + \$2 \times e^{0.04\times\frac{6}{12}} + \$3 \times e^{0.04\times\frac{3}{12}} + \$5 \times e^{0.04\times\frac{0}{12}}\right) \\
&= \$92.98.
\end{aligned}$$

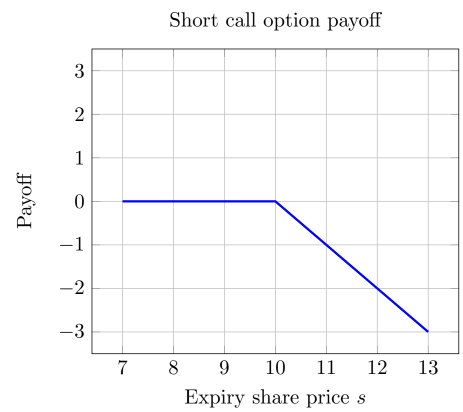
3.) (Long and short payoff diagrams) In this problem assume the underlier is a single share of a stock. Plot 6 payoff diagrams: for both long and short positions for a forward contract, a European put option, and a European call option. For concreteness, suppose \$10 is the strike price for both of the options and also the forward price for the forward contract. (Payoff diagrams plot gross revenue at maturity/expiration as a function of the underlier price, ignoring premium.)

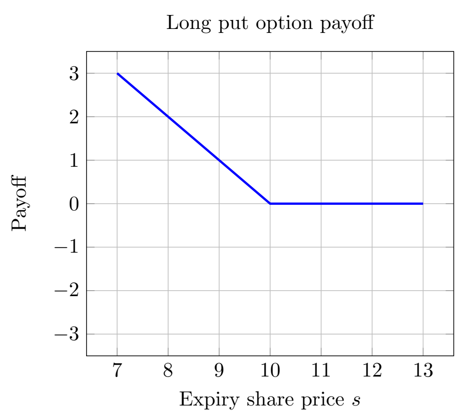
<u>Solution:</u> We plot pairs of diagrams below, according to long and short positions in each instrument.

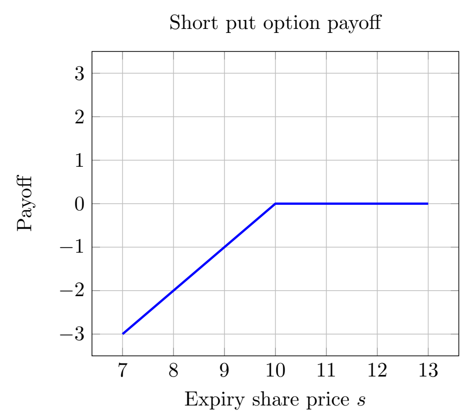












4.) (Put-call parity) Consider a forward contract with forward price F. Suppose S is the actual market price of the underlying asset at maturity, with maturity being T years.

(a) Using a rational pricing assumption with continuously compounded annual interest rate r, compute the *value* of this forward contract today, i.e., a reasonable premium today for entering into a long position in this contract.

(b) Using the results from the previous problem, use long and/or short positions in European call and/or put options to construct a payoff that matches a single long forward contract.

(c) Let P be the *value* of a put option today (e.g., the premium), and let C be the value of a call option. Derive a relation between the values of forward contract, put, and call options. (This relation is called *put-call parity*, and requires us to assume that the underliers are all the same, the strike price and forward price are the same, and that the expiry for all the contracts is the same.)

Solution:

(a) Under the rational pricing assumption, today's value of this forward contract would be tomorrow's profit discounted to today's value. Tomorrow's profit is $S - F$, and we must discount with continuously compounded interest with rate r and time T, so the value of this forward contract is $e^{-rT}(S - F)$.

(b) In order to reconstruct the payoff diagram for the forward contract, we can buy one long position in a European call option, and buy one short position in a European put option. I.e., the sum of the payoffs of a long call and a short put equals the payoff of a long forward contract.

(c) The principle here is that since a long forward contract has identical payoff to a combination of a long call and short put, then today's value of these two opportunities should be identical. Since we are shorting a put option, it's value (premium) is $-P$. Therefore, the sought relation is,

$$C - P = e^{-rT}(S - F)$$

It's more common to write this result as $C - P = S_0 - Fe^{-rT}$, where S_0 is *today's* asset price; these two expressions are equivalent under the no-arbitrage type of assumption $S = S_0 e^{rT}$.