Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 3 Solution

Simple portfolios

Due: Friday, Sept 13, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) Show the following:

(a) If $\boldsymbol{X} \in \mathbb{R}^n$ is a random variable and $\boldsymbol{a} \in \mathbb{R}^n$ is a(ny) deterministic vector, prove that $\text{Var}\,\langle \boldsymbol{a}, \boldsymbol{X} \rangle = \boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a}$, where $\boldsymbol{C} = \text{Cov}(\boldsymbol{X})$.

(b) (Portfolio weights) Consider a 4-security portfolio whose weights satisfy

$$\sum_{j=1}^{4} w_j = 1.$$

The weights form a 3-dimensional affine space. Determine an explicit 3-variable parameterization of this space, and provide a financial interpretation for the variables.

Solution:

(a) Define $Y \coloneqq \langle \boldsymbol{a}, \boldsymbol{X} \rangle$. The question asks us to compute a formula for $\text{Var}Y$. Another way to write this is $Y = \boldsymbol{a}^T \boldsymbol{X} = \boldsymbol{X}^T \boldsymbol{a}$. By linearity of the expectation, we have,

$$\mathbb{E}Y = \mathbb{E}\boldsymbol{a}^T \boldsymbol{X} = \boldsymbol{a}^T \mathbb{E}\boldsymbol{X} = \boldsymbol{a}^T \boldsymbol{\mu}, \qquad\qquad \boldsymbol{\mu} \coloneqq \mathbb{E}\boldsymbol{X}.$$

As an intermediate computation, we have,

$$Y - \mathbb{E}Y = \boldsymbol{a}^T \boldsymbol{X} - \boldsymbol{a}^T \boldsymbol{\mu} = \boldsymbol{a}^T \left(\boldsymbol{X} - \boldsymbol{\mu}\right) = \boldsymbol{a}^T \left(\boldsymbol{X} - \mathbb{E}\boldsymbol{X}\right).$$

Then the variance of Y is,

$$\begin{aligned}
\text{Var}Y &= \mathbb{E}\left[(Y - \mathbb{E}Y)(Y - \mathbb{E}Y)\right] \\
&= \mathbb{E}\left[(Y - \mathbb{E}Y)(Y - \mathbb{E}Y)^T\right] \\
&= \mathbb{E}\left[\boldsymbol{a}^T \left(\boldsymbol{X} - \mathbb{E}\boldsymbol{X}\right)\left(\boldsymbol{X} - \mathbb{E}\boldsymbol{X}\right)^T \boldsymbol{a}\right] \\
&= \boldsymbol{a}^T \mathbb{E}\left[\left(\boldsymbol{X} - \mathbb{E}\boldsymbol{X}\right)\left(\boldsymbol{X} - \mathbb{E}\boldsymbol{X}\right)^T\right] \boldsymbol{a} \\
&= \boldsymbol{a}^T (\text{Cov}\boldsymbol{X}) \boldsymbol{a} \\
&= \boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a}.
\end{aligned}$$

(b) The set of valid portfolio weights is determined by the matrix-vector equation,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \boldsymbol{w} = 1, \qquad\qquad \boldsymbol{w} = (1, 1, 1, 1)^T.$$

This is a linear equation whose general solution involves 3 free variables. One way to write this is as:

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 1 - w_1 - w_2 - w_3 \end{pmatrix},$$

where w_2, w_3, w_4 are the three free variables. To understand the roles that these variables play, we write the same solution as:

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 1 - w_1 - w_2 - w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + w_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, the variable w_1 corresponds to augmenting the portfolio with weights $(1, 0, 0, -1)^T$. In practice, this means that we purchase shares of security 1 by selling borrowed shares of security 4. (We are shorting security 4 to purchase security 1.) This intpretation is reversed is w_1 is negative. The other two free variables have similar interpretation:

- A positive w_2 corresponds to purchasing shares of security 2 by selling borrowed shares of security 4. (We are shorting security 4 to purchase security 2.) If w_2 is negative, we are shorting security 2 to purchase security 4.
- A positive w_3 corresponds to purchasing shares of security 3 by selling borrowed shares of security 4. (We are shorting security 4 to purchase security 3.) If w_3 is negative, we are shorting security 3 to purchase security 4.

Finally, note that there is no unique parameterization for the 3 free variables – the solution above is one choice, but there are many others. However, they will all correspond to shorting one or more securities in order to purchase one or more securities.

2.) (Portfolio risk) Consider a portfolio comprised of the sum of two different securities with per-share values $S_1(0) = 1$ and $S_2(0) = 1$, respectively. Assume an initial captial amount $V(0) = 1$ and that you are allowed to purchase fractional shares of each security. At time $t = 1$ the per-unit prices of the securities become random variables with mean and covariance given by,

$$\mathbb{E} \begin{pmatrix} S_1(1) \\ S_2(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \text{Cov} \begin{pmatrix} S_1(1) \\ S_2(1) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

(a) With this setup, show that the portfolio weights $\boldsymbol{w} = (w_1, w_2)^T$ coincide with the trading stategy $\boldsymbol{n} = (n_1, n_2)^T$.

(b) For a portfolio weight of $\boldsymbol{w} = \left(\frac{1}{4}, \frac{3}{4}\right)^T$, determine the mean and risk (standard deviation) of the time-1 portfolio value.

(c) Determine an initial portfolio weight vector $\boldsymbol{w}$ that minimizes the squared risk (variance) of the portfolio value at time 1.

(d) (**Math 6890 students only**) Assume we disallow short selling (negative portfolio weights). What portfolio weights maximize the average (mean) portfolio value at time 1, ignoring risk? Between the three portfolio weights identified in this and previous parts (along with the corresponding expected values and risks), describe how you might advise an investor to act.

Solution:

(a) The portfolio weight w_i (for $i = 1, 2$) is defined as the time-0 value $n_i S_i(0)$ of security i relative to the total capital $V(0)$:

$$w_i = \frac{n_i S_i(0)}{V(0)} \stackrel{S_i(0)=1, V(0)=1}{=} n_i.$$

Since this is true for $i = 1, 2$, then $\boldsymbol{w} = \boldsymbol{n}$.

(b) The value of the portfolio at time $t = 1$ is given by,

$$V(1) = \langle \boldsymbol{n}, \boldsymbol{S}(1) \rangle \stackrel{\text{part (a)}}{=} \langle \boldsymbol{w}, \boldsymbol{S}(1) \rangle .$$

Since the weights $\boldsymbol{w}$ are deterministic, the mean of the portfolio value is thus given by,

$$\mathbb{E} V(1) = \langle \boldsymbol{w}, \mathbb{E}\boldsymbol{S}(1) \rangle .$$

Using the problem-provided values of $\mathbb{E}\boldsymbol{S}(1)$ and $\boldsymbol{w}$, we compute:

$$\mathbb{E} V(1) = 2.75$$

To compute the variance, we identify a centered (mean-0) version of the time-1 value,

$$V(1) - \mathbb{E}V(1) = \langle \boldsymbol{w}, \boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \rangle = \boldsymbol{w}^T \left(\boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \right),$$

so that the variance of $V(1)$ is,

$$\begin{aligned} \text{Var} V(1) = \mathbb{E}\left(V(1) - \mathbb{E}V(1)\right)^2 &= \mathbb{E}\left[\boldsymbol{w}^T \left(\boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \right) \left(\boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \right)^T \boldsymbol{w} \right] \\ &= \boldsymbol{w}^T \mathbb{E}\left[\left(\boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \right) \left(\boldsymbol{S}(1) - \mathbb{E}\boldsymbol{S}(1) \right)^T \right] \boldsymbol{w} \\ &= \boldsymbol{w}^T \left[\text{Cov} \boldsymbol{S}(1) \right] \boldsymbol{w}. \end{aligned}$$

Using the problem-provided values of $\boldsymbol{w}$ and the covariance of $\boldsymbol{S}(1)$, we compute:

$$\text{Var} V(1) = 1.5,$$

and hence the risk is $\sqrt{\text{Var} V(1)} = \sqrt{3/2}$.

(c) Let $\boldsymbol{A} = \text{Cov}\boldsymbol{S}(1)$, then the variance $V(1)$ is a quadratic form of the portfolio weights $\boldsymbol{w}$ involving the symmetric, positive-definite matrix $\boldsymbol{A}$. Hence, we seek a solution to

$$\min \left\{ \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \;\middle|\; \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1 \right\}.$$

There are several ways to approach this optimization problem. Here is one strategy: Suppose that $\boldsymbol{w}$ is an arbitrary valid portfolio weight vector, i.e., that

$$w_1 + w_2 = 1.$$

Then due to the symmetry of $\boldsymbol{A}$, the variance of $V(1)$ is given by,

$$\begin{aligned}
\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} &= \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)^T \boldsymbol{A} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + w_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\
&= w_1^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2w_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= 8w_1^2 - 8w_1 + 3.
\end{aligned}$$

Hence, the variance is a quadratic function of w_1, and this is easily optimized via univariate calculus:

$$\underset{w_1 \in \mathbb{R}}{\operatorname{argmin}} (8w_1^2 - 8w_1 + 3) = \frac{1}{2},$$

Hence, the portfolio weights corresponding to the minimal variance are given by,

$$\boldsymbol{w} = \left(\frac{1}{2}, \ \frac{1}{2} \right)^T \implies \operatorname{Var} V(1) = 1.$$

For future reference we note that the mean of the portfolio value with this weighting is given by:

$$\mathbb{E} V(1) = \langle \boldsymbol{w}, \mathbb{E} \boldsymbol{S}(1) \rangle = 2.5.$$

(d) If we ignore risk and outlaw short-selling, then we are simply attempting to maximize the mean for non-negative portfolio weights:

$$\underset{w_1 \in [0,1]}{\operatorname{argmax}} \, \mathbb{E} V(1) = \underset{w_1 \in [0,1]}{\operatorname{argmax}} \, 2w_1 + 3(1 - w_1) = 0,$$

i.e., the portfolio $\boldsymbol{w} = (0, 1)^T$ produces a portfolio with time-1 statistics:

$$\mathbb{E} V(1) = \langle \boldsymbol{w}, \mathbb{E} \boldsymbol{S}(1) \rangle = 3, \qquad \operatorname{Var} V(1) = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = 3,$$

and this achieves the maximum possible average value.

To understand which of the three portfolio weightings is desirable, we summarize the results:

$$\begin{aligned}
\boldsymbol{w} = \left(\frac{1}{4}, \frac{3}{4} \right)^T &\implies (\mathbb{E} V(1), \operatorname{Var} V(1)) = (2.75, 1.5), \\
\boldsymbol{w} = \left(\frac{1}{2}, \frac{1}{2} \right)^T &\implies (\mathbb{E} V(1), \operatorname{Var} V(1)) = (2.5, 1), \\
\boldsymbol{w} = (0, 1)^T &\implies (\mathbb{E} V(1), \operatorname{Var} V(1)) = (3, 3).
\end{aligned}$$

There isn't a "correct" answer as to which of these is most desirable as each involves some risk. For a risk-averse investor, the third option is least desirable as the variation is the largest, but a risky investor might choose it. Between the first two options, the second option is the safest bet, but also involves slightly lower average return. An imperfect way to compare these options is to maximize some value that takes into account both the mean and the variance. For example, one such measure is $\mathbb{E}V(1) - \sqrt{\mathrm{Var}V(1)}$, which provides a number which ones hopes is exceeded with reasonably large probability. This computation for the 3 options above yields, respectively,

$$\begin{aligned}
\boldsymbol{w} = \left(\frac{1}{4}, \frac{3}{4}\right)^T &\Longrightarrow \mathbb{E}V(1) - \sqrt{\mathrm{Var}V(1)} \approx 1.53 \\
\boldsymbol{w} = \left(\frac{1}{2}, \frac{1}{2}\right)^T &\Longrightarrow \mathbb{E}V(1) - \sqrt{\mathrm{Var}V(1)} = 1.50 \\
\boldsymbol{w} = (0,1)^T &\Longrightarrow \mathbb{E}V(1) - \sqrt{\mathrm{Var}V(1)} \approx 1.27.
\end{aligned}$$

Hence, using this (imperfect!) metric, one might decide that the first option is the most attractive.

3.) (Hedging portfolio) Suppose we form a portfolio using two stocks with prices S_1 and S_2. Both stock shares have initial value $S_1(0) = S_2(0) = 1$. At time $t = 1$, the price of these shares is given by,

$$S_1(1) = 1 + a + X_1, \qquad\qquad S_2(1) = 1 + a + X_2,$$

where X_1 and X_2 are two random variables satisfying:

$$\begin{aligned}
&\mathbb{E}X_1 = 0, && \mathrm{Var}X_1 = \sigma_1^2 \\
&\mathbb{E}X_2 = 0, && X_2 = bX_1 + Z, && \mathrm{Var}Z = \sigma_2^2.
\end{aligned}$$

Above, a is some non-negative constant, and $b \in (-1, 0)$.

(a) Consider a portfolio with initial trading strategy $\boldsymbol{n} = \left(1, -\frac{1}{b}\right)^T$. If $V(t)$ is the total value (in dollars) of the portfolio, show that the return rate $R(1)$ defined as $R(1) = \frac{V(1)-V(0)}{V(0)}$ is $R(1) = a + Z/(1-b)$.

(b) Show that the variance of the return rate is always smaller than σ_2^2.

(c) Consider $a = 0.05$, $b = -0.5$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.2$. Compute the variance of $R(1)$.

Solution:

(a) The time-0 value of the portfolio is given by,

$$V(0) = \langle \boldsymbol{n}, \boldsymbol{S}(0) \rangle = S_1(0) - \frac{1}{b} S_2(0) \stackrel{c := 1 - 1/b}{=} c.$$

The time-1 value of the portfolio is given by

$$\begin{aligned}
V(1) = \langle \boldsymbol{n}, \boldsymbol{S}(1) \rangle = S_1(1) - \frac{1}{b} S_2(1) &= 1 + a + X_1 - \frac{1}{b} - \frac{a}{b} - \frac{1}{b} X_2 \\
&= c + ac + X_1 - \frac{b}{b} X_1 - \frac{1}{b} Z \\
&= c(1 + a) - \frac{1}{b} Z
\end{aligned}$$

Hence, the return $R(1)$ is given by,

$$R(1) = \frac{V(1) - V(0)}{V(0)} = \frac{1}{c}\left(ac - \frac{1}{b}Z\right) = a - \frac{1}{bc}Z = a + \frac{Z}{1-b}.$$

(b) The scalars a, b are deterministic, so the variance of $R(1)$ is,

$$\mathrm{Var} R(1) = \frac{\mathrm{Var} Z}{|1-b|^2}.$$

Since $b \in (-1, 0)$, then $|1-b| \in (1, 2)$, and hence

$$\mathrm{Var} R(1) \leq \mathrm{Var} Z.$$

(c) Using a computation from the previous part, we compute:

$$\mathrm{Var} R(1) = \frac{\mathrm{Var} Z}{|1-b|^2}. = \frac{0.2^2}{|1+0.5|^2} \approx 0.0178$$