Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2024
Homework 3
Simple portfolios

Due: Friday, Sept 13, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) Show the following:

(a) If $\boldsymbol{X} \in \mathbb{R}^n$ is a random variable and $\boldsymbol{a} \in \mathbb{R}^n$ is a(ny) deterministic vector, prove that $\text{Var}\left\langle \boldsymbol{a}, \boldsymbol{X} \right\rangle = \boldsymbol{a}^T \boldsymbol{C} \boldsymbol{a}$, where $\boldsymbol{C} = \text{Cov}(\boldsymbol{X})$.

(b) (Portfolio weights) Consider a 4-security portfolio whose weights satisfy

$$\sum_{j=1}^{4} w_j = 1.$$

The weights form a 3-dimensional affine space. Determine an explicit 3-variable parameterization of this space, and provide a financial interpretation for the variables.

2.) (Portfolio risk) Consider a portfolio comprised of the sum of two different securities with per-share values $S_1(0) = 1$ and $S_2(0) = 1$, respectively. Assume an initial captial amount $V(0) = 1$ and that you are allowed to purchase fractional shares of each security. At time $t = 1$ the per-unit prices of the securities become random variables with mean and covariance given by,

$$\mathbb{E}\begin{pmatrix} S_1(1) \\ S_2(1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \text{Cov}\begin{pmatrix} S_1(1) \\ S_2(1) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

(a) With this setup, show that the portfolio weights $\boldsymbol{w} = (w_1, w_2)^T$ coincide with the trading stategy $\boldsymbol{n} = (n_1, n_2)^T$.

(b) For a portfolio weight of $\boldsymbol{w} = \left(\frac{1}{4}, \frac{3}{4}\right)^T$, determine the mean and risk (standard deviation) of the time-1 portfolio value.

(c) Determine an initial portfolio weight vector $\boldsymbol{w}$ that minimizes the squared risk (variance) of the portfolio value at time 1.

(d) (**Math 6890 students only**) Assume we disallow short selling (negative portfolio weights). What portfolio weights maximize the average (mean) portfolio value at time 1, ignoring risk? Between the three portfolio weights identified in this and previous parts (along with the corresponding expected values and risks), describe how you might advise an investor to act.

3.) (Hedging portfolio) Suppose we form a portfolio using two stocks with prices S_1 and S_2. Both stock shares have initial value $S_1(0) = S_2(0) = 1$. At time $t = 1$, the price of these shares is given by,

$$S_1(1) = 1 + a + X_1, \qquad S_2(1) = 1 + a + X_2,$$

where X_1 and X_2 are two random variables satisfying:

$$\mathbb{E}X_1 = 0, \qquad \text{Var}X_1 = \sigma_1^2$$
$$\mathbb{E}X_2 = 0, \qquad X_2 = bX_1 + Z, \qquad \text{Var}Z = \sigma_2^2.$$

Above, a is some non-negative constant, and $b \in (-1, 0)$.

(a) Consider a portfolio with initial trading strategy $\boldsymbol{n} = \left(1, -\frac{1}{b}\right)^T$. If $V(t)$ is the total value (in dollars) of the portfolio, show that the return rate $R(1)$ defined as $R(1) = \frac{V(1)-V(0)}{V(0)}$ is $R(1) = a + Z/(1-b)$.

(b) Show that the variance of the return rate is always smaller than σ_2^2.

(c) Consider $a = 0.05$, $b = -0.5$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.2$. Compute the variance of $R(1)$.