Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 4 Solutions

2-security Markowitz portfolios

Due: Friday, Sept 20, 2024

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1.) (Markowitz 2-security portfolios) Consider a 2-security portfolio having per-unit asset prices $S_1(t)$ and $S_2(t)$. Assume the following statistics for these assets:

$$
\begin{aligned}
&S_1(0) = 100 \text{ (with probability 1)} \qquad && \mathbb{E}S_1(1) = 120, \qquad && \sqrt{\mathrm{Var}\, S_1(1)} = 20 \\
&S_2(0) = 50 \text{ (with probability 1)} \qquad && \mathbb{E}S_2(1) = 75, \qquad && \sqrt{\mathrm{Var}\, S_2(1)} = 40,
\end{aligned}
$$

along with $\mathrm{Cov}(S_1(1), S_2(1)) = -500$.

(a) Show that the return rates $\boldsymbol{R}$ of the individual securities in this setup have statistics,

$$
\mathbb{E}\boldsymbol{R}(1) = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \qquad \mathrm{Cov}\boldsymbol{R}(1) = \begin{pmatrix} 0.04 & -0.1 \\ -0.1 & 0.64 \end{pmatrix}
$$

(b) Compute the minimum-risk portfolio for a general expected return rate μ_P.

<u>**Solution**</u>:

(a) We translate the given statistics into corresponding statistics for return rates. We know that such portfolios have the return,

$$
R(t) = \langle \boldsymbol{w}, \boldsymbol{R} \rangle, \qquad R_i(t) = \frac{S_i(t) - S_i(0)}{S_i(0)},
$$

for $i = 1, 2$, where $\boldsymbol{w}$ contains the unknown portfolio weights. Using standard properties of first- and second-order statistics, we have:

$$
\begin{aligned}
\mu_1 &= \mathbb{E}R_1(1) = \frac{\mathbb{E}S_1(1)}{S_1(0)} - 1 = 0.2, \qquad \sigma_1 = \sqrt{\mathrm{Var}R_1(1)} = \frac{\sqrt{\mathrm{Var}S_1(1)}}{S_1(0)} = 0.2, \\
\mu_2 &= \mathbb{E}R_2(1) = \frac{\mathbb{E}S_2(1)}{S_2(0)} - 1 = 0.5, \qquad \sigma_2 = \sqrt{\mathrm{Var}R_2(1)} = \frac{\sqrt{\mathrm{Var}S_2(1)}}{S_2(0)} = 0.8
\end{aligned}
$$

The covariance satisfies similar properties:

$$
\mathrm{Cov}(R_1(1), R_2(1)) = \mathrm{Cov}\left(\frac{S_1(1)}{S_1(0)}, \frac{S_2(1)}{S_2(0)} \right) = \frac{1}{S_1(0)S_2(0)} \mathrm{Cov}(S_1(1), S_2(1)) = -0.1.
$$

Hence, we have

$$
\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}(1) = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \qquad \boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R}(1)) = \begin{pmatrix} 0.2^2 & -0.1 \\ -0.1 & 0.8^2 \end{pmatrix},
$$

as desired.

(b) The Markowitz portfolio optimization for this setup is,

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \quad \text{subject to} \quad \begin{aligned} \langle \boldsymbol{w}, \boldsymbol{1} \rangle &= 1 \text{ and} \\ \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle &= \mu_P \end{aligned}$$

Since this is a 2-security portfolio, the two linear constraints will determine $\boldsymbol{w}$ without optimization:

$$\left.\begin{aligned} w_1 + w_2 &= 1 \\ 0.2 w_1 + 0.5 w_2 &= \mu_P \end{aligned}\right\} \Longrightarrow \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 - 10\mu_P \\ 10\mu_P - 2 \end{pmatrix} = \frac{1}{3} \left[\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \mu_P \begin{pmatrix} -10 \\ 10 \end{pmatrix} \right]$$

Hence, given μ_P, the weights above prescribe the optimal (minimal) risk portfolio.

2.) (Short-selling in 2-security portfolios) Consider a 2-security portfolio with asset 1 and asset 2. Assume the time-1 asset return rates have means μ_1 and μ_2, respectively, and that $\mu_1 \neq \mu_2$. Assume that $\mu_1 < \mu_2$ (this assumption is without loss). For a Markowitz portfolio with target return rate μ_P, show the following:

(a) $\mu_P \in [\mu_1, \mu_2]$ if and only if the 2-security portfolio involves long (or zero) positions in both securities.

(b) $\mu_P < \mu_1$ if and only if the 2-security portfolio involves a long position in asset 1 and a short position in asset 2.

(c) $\mu_P > \mu_2$ if and only if the 2-security portfolio involves a long position in asset 2 and a short position in asset 1.

<u>**Solution**</u>:

(a) For a 2-security portfolio with $\mu_1 \neq \mu_2$ and target return μ_P, the portfolio weights are entirely determined by the linear constraints:

$$\begin{aligned} w_1 + w_2 &= 1 \\ w_1 \mu_1 + w_2 \mu_2 &= \mu_P. \end{aligned}$$

The solution to this linear system is,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{\mu_2 - \mu_1} \begin{pmatrix} \mu_2 - \mu_P \\ \mu_P - \mu_1 \end{pmatrix}$$

Note that $\mu_2 - \mu_1 > 0$, so that we have the implications,

$$\begin{aligned} \mu_P \in [\mu_1, \mu_2] &\iff \mu_2 - \mu_P \geq 0 \iff w_1 \geq 0 \\ \mu_P \in [\mu_1, \mu_2] &\iff \mu_P - \mu_1 \geq 0 \iff w_2 \geq 0 \end{aligned}$$

I.e., $\mu_P \in [\mu_1, \mu_2]$ is equivalent to long or zero positions (non-negative weights) in both asset 1 and 2.

(b) Starting from the expression for the weights derived in part (a), we observe,

$$\begin{aligned} \mu_P < \mu_1 &\iff \mu_2 - \mu_P \geq 0 \iff w_1 \geq 0 \\ \mu_P < \mu_1 &\iff \mu_P - \mu_1 < 0 \iff w_2 < 0 \end{aligned}$$

i.e., that $\mu_P < \mu_1$ is equivalent to a long position (non-negative weight) in asset 1 and a short position (negative weight) in asset 2.

(c) Starting from the expression for the weights derived in part (a), we observe,

$$\mu_P > \mu_2 \iff \mu_2 - \mu_P < 0 \iff w_1 < 0$$
$$\mu_P > \mu_2 \iff \mu_P - \mu_1 \geq 0 \iff w_2 \geq 0$$

i.e., that $\mu_P > \mu_2$ is equivalent to a long position (non-negative weight) in asset 2 and a short position (negative weight) in asset 1.

3.) (Arbitrage in portfolios) Consider a 2-security portfolio consisting of asset 1 and asset 2. Assume the time-1 asset return rates R_1 and R_2 have mean and standard deviation (μ_1, σ_1) and (μ_2, σ_2), respectively. Assume that $\sigma_1 + \sigma_2 > 0$, i.e., that at least one security is random.

(a) Recall that the Pearson correlation coefficient between R_1 and R_2 is defined as $\rho := \text{Cov}(R_1, R_2)/(\sigma_1\sigma_2)$. If $\rho = -1$, explicitly construct a zero-risk portfolio using a non-trivial linear combination of assets 1 and 2.

(b) Using the previous result, give a necessary and sufficient condition involving the statistics above that ensures that an arbitrage, i.e., a riskless (with strictly positive) profit strategy, exists.

(c) **(Math 6890 students only)** Extend part of this to the N-security case: Show that if the covariance matrix of the individual security return rates is <u>not</u> positive-definite, instead only of rank $N-1$, then a riskless security can be constructed, and provide (perhaps opaque but symbolically explicit) conditions on the security statistics that ensure that this riskless security can be used for arbitrage. Your conditions may involve eigenvalues/vectors of the covariance matrix.

<u>**Solution**</u>:

(a) If $\rho = -1$, then the covariance matrix of $\boldsymbol{R}$ can be written as,

$$\boldsymbol{A} := \text{Cov}(\boldsymbol{R}) = \begin{pmatrix} \sigma_1^2 & \text{Cov}(R_1, R_2) \\ \text{Cov}(R_1, R_2) & \sigma_2^2 \end{pmatrix} \overset{\rho=-1}{=} \begin{pmatrix} \sigma_1^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

The risk of the portfolio is $\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w}$; since $\boldsymbol{A}$ is symmetric, then $\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = 0$ implies that $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}$, i.e., that $\boldsymbol{w}$ is a vector in the nullspace of $\boldsymbol{A}$. The explicit form of $\boldsymbol{A}$ above shows that one such vector is given by,

$$\boldsymbol{w} = \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix}.$$

To make this vector a valid portfolio weight vector, we normalize appropriately:

$$\boldsymbol{w} = \frac{1}{\sigma_1 + \sigma_2} \begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix},$$

whose components sum to unity (making it a valid portfolio weight) and is well-defined since $\sigma_1 + \sigma_2 > 0$. Hence, this forms a zero-risk portfolio.

(b) In order for the portfolio identified in the previous part to be an arbitrage, its mean must be strictly positive. The mean of the portfolio above is,

$$\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle = \frac{\sigma_2\mu_1 + \sigma_1\mu_2}{\sigma_1 + \sigma_2}.$$

Since the denominator is positive, the number above is positive if and only if

$$\sigma_2\mu_1 + \sigma_1\mu_2 > 0.$$

Therefore, under this condition, the portfolio weight vector identified in part (a) is a riskless portfolio with strictly positive mean, i.e., corresponds to an arbitrage.

(c) In the N-security case, if $\boldsymbol{A} = \text{Cov}(\boldsymbol{R})$ is *not* positive-definite, then there exists some non-zero vector $\boldsymbol{v}$ such that,

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0},$$

so that $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = 0$. Note that since $\text{rank}(\boldsymbol{A}) = N - 1$, then $\boldsymbol{v}$ is unique up to a consatnt. In order to be able to normalize $\boldsymbol{v}$ so that it's a valid portfolio weight vector, we must have,

$$\langle \boldsymbol{v}, \boldsymbol{1} \rangle \neq 0.$$

Assuming this, then

$$\boldsymbol{w} := \frac{1}{\langle \boldsymbol{v}, \boldsymbol{1} \rangle}\boldsymbol{v},$$

is a valid portfolio weight vector corresponding to a riskless portfolio. This is the only such portfolio since $\boldsymbol{v}$ is unique up to a constant. In order for it to be an arbitrage, it must have positive mean:

$$\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle > 0,$$

where $\boldsymbol{\mu}$ is the mean of the individual securities. Using the expressions we've derived above, this is equivalent to:

$$\frac{\langle \boldsymbol{v}, \boldsymbol{\mu} \rangle}{\langle \boldsymbol{v}, \boldsymbol{1} \rangle} > 0, \qquad \langle \boldsymbol{v}, \boldsymbol{1} \rangle \neq 0$$

These are conditions that explicitly involve $\boldsymbol{\mu}$, and implicitly involve entries of the second-order statistics $\text{Cov}(\boldsymbol{R})$ since $\boldsymbol{v}$ is an eigenvector of $\text{Cov}(\boldsymbol{R})$.

4.) Consider a Markowitz 2-security portfolio with a given terminal time positive-definite covariance $\text{Cov}(\boldsymbol{R})$ and terminal time mean $\boldsymbol{\mu}$. Assume that,

$$\mu_1 = \mu_2.$$

(a) Show that any Markowitz portfolio must have expected return μ_P given by $\mu_P = \mu_1 = \mu_2$.

(b) For the covariance matrix,

$$\text{Cov}(\boldsymbol{R}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

compute both the optimal Markowitz portfolio and its corresponding risk.

Solution:

(a) For notational simplicity, we'll let $\mu = \mu_1 = \mu_2$. The weights of the portfolio must satisfy,

$$w_1 + w_2 = 1$$
$$\mu w_1 + \mu w_2 = \mu_P.$$

If $\mu = 0$, then clearly $\mu_P = 0 = \mu$. If $\mu \neq 0$, then the second equation is equivalent to,

$$w_1 + w_2 = \frac{\mu_P}{\mu}.$$

In order for this to be consistent with the first equation, we must have $\mu_P = \mu$. (Otherwise the weight constraints are inconsistent and no valid portfolio exists.) Hence, no matter what value of μ, we must have $\mu_P = \mu$.

(b) With our $\mu = \mu_1 = \mu_2$ setup, then the two linear constraints on a Markowitz portfolio are simply the single condition,

$$w_1 + w_2 = 1.$$

Hence, the squared risk of the portfolio with $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$ is,

$$\sigma_P^2 = \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix}^T \boldsymbol{A} \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix} = \boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 + \left(2\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1\right) w_1 + \left(\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1\right) w_1^2,$$

where

$$\boldsymbol{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We directly compute:

$$\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_0 = 2, \qquad 2\boldsymbol{v}_0^T \boldsymbol{A} \boldsymbol{v}_1 = -6, \qquad \boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1 = 6,$$

so the risk squared reads,

$$\sigma_P^2 = 6w_1^2 - 6w_1 + 2.$$

Using univariate calculus to compute critical points, we conclude that the minimum of σ_P^2 occurs when,

$$w_1 = \frac{1}{2} \implies \boldsymbol{w} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

The (minimal) squared risk for this value of $\boldsymbol{w}$ is,

$$\sigma_P^2\big|_{w_1 = 0.5} = \frac{1}{2}.$$

Hence, the minimal squared risk is $\frac{1}{\sqrt{2}}$.