Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 4

2-security Markowitz portfolios

Due: Friday, Sept 20, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) (Markowitz 2-security portfolios) Consider a 2-security portfolio having per-unit asset prices $S_1(t)$ and $S_2(t)$. Assume the following statistics for these assets:

$$S_1(0) = 100 \text{ (with probability 1)} \qquad \mathbb{E}S_1(1) = 120, \qquad \sqrt{\operatorname{Var} S_1(1)} = 20$$
$$S_2(0) = 50 \text{ (with probability 1)} \qquad \mathbb{E}S_2(1) = 75, \qquad \sqrt{\operatorname{Var} S_2(1)} = 40,$$

along with $\operatorname{Cov}(S_1(1), S_2(1)) = -500$.

(a) Show that the return rates $\boldsymbol{R}$ of the individual securities in this setup have statistics,

$$\mathbb{E}\boldsymbol{R}(1) = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \qquad \operatorname{Cov}\boldsymbol{R}(1) = \begin{pmatrix} 0.04 & -0.1 \\ -0.1 & 0.64 \end{pmatrix}$$

(b) Compute the minimum-risk portfolio for a general expected return rate μ_P.

2.) (Short-selling in 2-security portfolios) Consider a 2-security portfolio with asset 1 and asset 2. Assume the time-1 asset return rates have means μ_1 and μ_2, respectively, and that $\mu_1 \neq \mu_2$. Assume that $\mu_1 < \mu_2$ (this assumption is without loss). For a Markowitz portfolio with target return rate μ_P, show the following:

(a) $\mu_P \in [\mu_1, \mu_2]$ if and only if the 2-security portfolio involves long (or zero) positions in both securities.

(b) $\mu_P < \mu_1$ if and only if the 2-security portfolio involves a long position in asset 1 and a short position in asset 2.

(c) $\mu_P > \mu_2$ if and only if the 2-security portfolio involves a long position in asset 2 and a short position in asset 1.

3.) (Arbitrage in portfolios) Consider a 2-security portfolio consisting of asset 1 and asset 2. Assume the time-1 asset return rates R_1 and R_2 have mean and standard deviation (μ_1, σ_1) and (μ_2, σ_2), respectively. Assume that $\sigma_1 + \sigma_2 > 0$, i.e., that at least one security is random.

(a) Recall that the Pearson correlation coefficient between R_1 and R_2 is defined as $\rho \coloneqq \operatorname{Cov}(R_1, R_2)/(\sigma_1 \sigma_2)$. If $\rho = -1$, explicitly construct a zero-risk portfolio using a non-trivial linear combination of assets 1 and 2.

(b) Using the previous result, give a necessary and sufficient condition involving the statistics above that ensures that an arbitrage, i.e., a riskless (with strictly positive) profit strategy, exists.

(c) **(Math 6890 students only)** Extend part of this to the N-security case: Show that if the covariance matrix of the individual security return rates is <u>not</u> positive-definite, instead only of rank $N-1$, then a riskless security can be constructed, and provide (perhaps opaque but symbolically explicit) conditions on the security statistics that ensure that this riskless security can be used for arbitrage. Your conditions may involve eigenvalues/vectors of the covariance matrix.

4.) Consider a Markowitz 2-security portfolio with a given terminal time positive-definite covariance $\mathrm{Cov}(\boldsymbol{R})$ and terminal time mean $\boldsymbol{\mu}$. Assume that,

$$\mu_1 = \mu_2.$$

(a) Show that any Markowitz portfolio must have expected return μ_P given by $\mu_P = \mu_1 = \mu_2$.

(b) For the covariance matrix,

$$\mathrm{Cov}\left(\boldsymbol{R}\right) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

compute both the optimal Markowitz portfolio and its corresponding risk.