Department of Mathematics, University of Utah
Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2024
Homework 5
N-security Markowitz Portfolios

Due: Friday, Sept 27, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) (Markowitz 2-security efficient frontier) From the last assignment, recall a 2-security setup with the following statistics:

$$\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R} = \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix}, \qquad \boldsymbol{A} = \mathrm{Cov}\boldsymbol{R} = \begin{pmatrix} 0.04 & -0.1 \\ -0.1 & 0.64 \end{pmatrix}$$

(a) On an expected return rate vs. risk figure, plot the set of optimal (minimum-risk) portfolios and identify the efficient frontier.

(b) An investor seeks to utilize this optimized portfolio corresponding to the expected return rate of $\mu_P = 15\%$. Would you recommend the corresponding portfolio to this person?

2.) (Petters & Dong, Problem 3.13, *Three Securities*) Suppose that you have \$5,000 to invest in stocks 1, 2, and 3 with current prices

$$\boldsymbol{S}(0) = \begin{pmatrix} \$10.20 \\ \$53.75 \\ \$30.45 \end{pmatrix},$$

along with time-1 expected return vector and covariance matrix given by,

$$\boldsymbol{\mu} = \begin{pmatrix} 0.10 \\ 0.15 \\ 0.075 \end{pmatrix}, \qquad \mathrm{Cov}(\boldsymbol{R}) = \boldsymbol{A} = \begin{pmatrix} 0.03 & -0.04 & 0.02 \\ -0.04 & 0.08 & -0.04 \\ 0.02 & -0.04 & 0.04 \end{pmatrix}.$$

For example, stock 3 has a volatility of $\sigma_3 = 20\%$ and expected return rate of $\mu_3 = 7.5\%$. Answer the following, using software if desired.

(a) Determine the weights needed to create the global minimum-variance portfolio of these three stocks.

(b) Create an efficient portfolio with an expected return rate of 18%. Explicitly state the number of shares one must hold for each stock and how you fund each position. State the portfolio risk and compare it with the maximum risk among the individual stocks.

3.) (N-security global minimizing mean) On slides D10-S05(b) of the lecture slides, an explicit formula for the mean μ_G of the global variance-minimizing N-security Markowtiz portfolio is provided. Simplify this formula and show that μ_G has the more direct expression:

$$\mu_G = \frac{b}{a} = \frac{\mathbf{1}^T \boldsymbol{A}^{-1}\boldsymbol{\mu}}{\mathbf{1}^T \boldsymbol{A}^{-1}\mathbf{1}},$$

where a, b refers to notation used on slide D10-S05. Use the formula above to justify why the assumption,

$$\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0,$$

is a reasonable practical assumption to make.

4.) (**Math 6890 students only**) (N-security portfolios) Consider the Lagrange multipliers methods for computing the risk-optimal N-security Markowitz portfolio (as done in class and also in the book). With this method, λ_1 corresponds to the constraint $\langle \boldsymbol{w}, \mathbf{1} \rangle = \mathbf{1}$, and λ_2 corresponds to the constraint $\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P$.

(a) Show that if we choose the global variance-minimizing portfolio, then this corresponds to $\lambda_2 = 0$. (The formula $\mu_G = b/a$ from the previous problem can be very helpful here.)

(b) Suppose $\lambda_1 = 0$, and assume $\mathbf{1}^T \boldsymbol{A}^{-1} \boldsymbol{\mu} > 0$. Show that the mean of this portfolio is given by,

$$\mu_P = \frac{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \boldsymbol{\mu}}{\boldsymbol{\mu}^T \boldsymbol{A}^{-1} \mathbf{1}},$$

and also show that this corresponds to an efficient portfolio. This portfolio is called the *diversified portfolio*. (It may be useful to recall that our general Markowitz portfolio setup assumes that $\mathbf{1}$ and $\boldsymbol{\mu}$ are not parallel vectors.)