Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 7 Solutions

The Binomial Pricing Model

Due: Friday, Oct 25, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) Consider an $n = 100$-period Binomial Pricing Model with $(p, u, d) = (0.35, 1.2, 0.9)$ and an initial value of $S_0 = 100$.

(a) What is the maximal value of S_{100} under this model? The minimal value?

(b) Compute the probability that $S_{100} \geq 100$.

Solution:

(a) The maximum value of S_{100} corresponds to 100 instances of the inter-period gross return rate being $u = 1.2$. The corresponding value would be,

$$S_{100} = 100(1.2)^{100} = 8.28 \times 10^9$$

Similarly, the minimal value corresponds to 100 instances of $d = 0.9$ as the inter-period gross return:

$$S_{100} = 100(0.9)^{100} = 2.66 \times 10^{-3}$$

(b) Since $100 = S_0$, then the question is identical to determining the probability that $S_n \geq S_0$. Note that the gross return is given by,

$$\frac{S_n}{S_0} = u^Y d^{100-Y},$$

where $Y \sim \text{Binomial}(100, 0.35)$. By direct computation, we find that

$$u^{36} d^{100-36} < 1, \qquad\qquad u^{37} d^{100-37} > 1.$$

(One can also solve for y in $(u/d)^y d^{100} = 1$, which yields $y \approx 36.624$, implying that $y = 36$ and $y = 37$ satisfy the upper and lower inequalities above, respectively.) Therefore,

$$\Pr\left(\frac{S_n}{S_0} \geq 1\right) = \Pr\left(Y \geq 37\right).$$

Hence, we need to compute this probability for a Binomial$(100, 0.35)$ random variable. Since,

$$\Pr(Y = k) = \binom{100}{k} 0.35^k (1 - 0.35)^{100-k},$$

then,

$$\Pr\left(\frac{S_n}{S_0} \geq 1\right) = \sum_{k=37}^{100} \binom{100}{k} 0.35^k (1 - 0.35)^{100-k} \approx 0.373.$$

2.) Let $X \sim \text{Bernoulli}(p)$, and let $\{X_i\}_{i=1}^n$ be n iid copies of X. Let $Y = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$. Throughout this problem, let $a > 0$ be a deterministic constant.

(a) Compute $\mathbb{E}a^X$ with $a > 0$ a deterministic constant.

(b) If V and W are two independent random variables, then $\mathbb{E}(VW) = (\mathbb{E}V)(\mathbb{E}W)$. Use this to compute $\mathbb{E}a^Y$.

(c) Compute the variance of a^Y.

(d) Apply these facts to the Binomial Pricing Model with parameters (p, u, d): with $S_n = S_0 e^L$, where $L = \sum_{i=1}^n L_i = \sum_{i=1}^n \log G_i$ is the log-return, show that,

$$\mathbb{E}\frac{S_n}{S_0} = (pu + (1-p)d)^n,$$
$$\text{Var}\frac{S_n}{S_0} = \left(pu^2 + (1-p)d^2\right)^n - (pu + (1-p)d)^{2n}$$

<u>**Solution**</u>:

(a) Since X has distribution,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p, \end{cases}$$

then we can directly compute,

$$\mathbb{E}a^X = pa^1 + (1-p)a^0 = pa + 1 - p.$$

(b) We write a^Y as,

$$a^Y = a^{\sum_{j=1}^n X_j} = \prod_{j=1}^N a^{X_j}.$$

Using the independent property mentioned in the problem, and since the $\{X_j\}_{j=1}^n$ are independent, we have,

$$\mathbb{E}a^Y = \prod_{j=1}^n \mathbb{E}a^{X_j} = \left(\mathbb{E}a^X\right)^n,$$

where the last equality uses the fact that $\{X_j\}_{j=1}^n$ and X are identically distributed. Using this along with part (a), we have:

$$\mathbb{E}a^Y = (pa + 1 - p)^n$$

(c) Letting $\mu = \mathbb{E}a^Y$ (which we computed above), we write the variance as,

$$\begin{aligned}
\text{Var } a^Y &= \mathbb{E}\left(a^Y - \mu\right)^2 \\
&= \mathbb{E}\left(a^Y\right)^2 - 2\mathbb{E}\left(\mu a^Y\right) + \mu^2 \\
&= \mathbb{E}\left(a^Y\right)^2 - 2\mu^2 + \mu^2 \\
&= \mathbb{E}(a^2)^Y - \mu^2.
\end{aligned}$$

We already have an expression for μ; to compute the expression for the first term, we note that it's the same as $\mathbb{E}a^Y$, but with the replacement $a \leftarrow a^2$. Therefore, we have:

$$\text{Var } a^Y = \left(pa^2 + 1 - p\right)^n - (pa + 1 - p)^{2n}$$

(d) For the Binomial Pricing Model, recall that the log-return L_j for time-step j is,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Hence, L_j can be written as,

$$L_j = \log d + X_j (\log u - \log d) = \log d + X_j \log \frac{u}{d},$$

where $X_j \sim \text{Bernoulli}(p)$. In particular, this implies that

$$\begin{aligned} e^L = e^{\sum_{j=1}^n L_j} &= e^{n \log d + \log \frac{u}{d} \sum_{j=1}^n X_j} \\ &= d^n \left(e^{\log \frac{u}{d}}\right)^{\sum_{j=1}^n X_j} \\ &= d^n \left(\frac{u}{d}\right)^Y \end{aligned}$$

Then using the results from the previous parts with $a = \frac{u}{d}$, along with $\mathbb{E}cW = c\mathbb{E}W$ and $\text{Var}(cW) = c^2 \text{Var} W$ for a constant c, we have:

$$\begin{aligned} \mathbb{E}e^L &= d^n \left(p\frac{u}{d} + 1 - p\right)^n = (pu + (1-p)d)^n \\ \text{Var } e^L &= d^{2n} \left[\left(p\frac{u^2}{d^2} + 1 - p\right)^n - \left(p\frac{u}{d} + 1 - p\right)^{2n} \right] \\ &= \left(pu^2 + (1-p)d^2\right)^n - (pu + (1-p)d)^{2n}, \end{aligned}$$

as desired.