Department of Mathematics, University of Utah

Introduction to Mathematical Finance

MATH 5760/6890 – Section 001 – Fall 2024

Homework 8 Solutions

More on Binomial Pricing Models

Due: Friday, November 1, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) (Petters & Dong, Problem 5.5) For an n-period binomial tree, let N_U be the number of security price upticks from time t_0 to t_n. Explain why N_U is a binomial random variable. What are its expected value and variance if the tree has 40 steps and the uptick probability is 60%?

Solution: We recall that each up/downtick is essentially a coin flip, with an uptick happening with probability p and a downtick with probability $1 - p$. Each up/downtick is independent of all the others. Therefore, for an n-period model, the total number of upticks N_U is given by,

$$N_U = \sum_{j=1}^{n} X_j, \qquad\qquad X_j \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$$

Hence N_U counts the number of $+1$'s encountered in the time-n process $\{X_j\}_{j=1}^{n}$ with success $(+1)$ probability p. By definition, this is a Binomial(n, p) random variable.

If we have a model with 40 steps, this corresponds to $n = 40$, and the uptick probability of 60% means that $p = 0.6$. Hence, N_U is a Binomial$(40, 0.6)$ random variable. A general Binomial(n, p) random variable X has the following statistics, as determined in class:

$$\mathbb{E}X = np, \qquad\qquad \text{Var}X = np(1-p).$$

Hence, with $(n, p) = (40, 0.6)$, then N_U has the following statistics:

$$\mathbb{E}N_U = 24, \qquad\qquad \text{Var}N_U = 9.6$$

2.) Consider an n-period Binomial Pricing Model for an asset over the time interval $t \in [0, T]$ with $d = 1/u < 1$. (This is a special type of recombination condition.)

(a) Show that the expected value of the gross return, $\mathbb{E}\frac{S_n}{S_0}$, is given by $(pu + (1-p)/u)^n$. (You may use results from previous homework assignments if desired.)

(b) Suppose that from historical data we compute a T-time expected return rate r for the asset:

$$\mathbb{E}S_n = S_0(1 + r),$$

where r is a deterministic, positive constant $r > 0$. Show that in order for S_n under the given binomial pricing model to achieve an expected gross return rate $(1 + r)$, then u must satisfy,

$$u = \frac{1}{2p}\left(e^{\mu} \pm \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

where $\mu = \frac{\log(1+r)}{n}$.

(c) Show that $e^{2\mu} > 4p(1-p)$, and hence there are always two real values of u above.

(d) Show that if we choose the minus option in formula with $\pm$ above, then $u < 1$, and hence the only valid choice is the plus option.

Solution:

(a) We have that

$$\frac{S_n}{S_0} = \prod_{j=1}^{n} G_j,$$

where G_j is a shifted Bernoulli random variable:

$$G_j = \begin{cases} u, & \text{with probability } p, \\ \frac{1}{u}, & \text{with probability } 1-p \end{cases}$$

Hence, taking expectations and noting that the G_j are independent, we have,

$$\mathbb{E}\frac{S_n}{S_0} = \mathbb{E}\left(\prod_{j=1}^{n} G_j\right) = \prod_{j=1}^{n} \mathbb{E}G_j = \prod_{j=1}^{n} (pu + (1-p)/u) = \left(pu + \frac{1-p}{u}\right)^n,$$

as desired.

(b) We have computed the expected gross return rate in the previous part, and this must match $(1+r)$:

$$\left(pu + \frac{1-p}{u}\right)^n = 1 + r,$$

Raising both sides to the $1/n$ power (and using the definition of μ) yields,

$$pu + \frac{1-p}{u} = e^{\mu}.$$

Multiplying by u on both sides yields the quadratic condition:

$$pu^2 - e^{\mu}u + (1-p) = 0.$$

The two roots of this equation are found from the quadratic formula:

$$u = \frac{1}{2p}\left(e^{\mu} \pm \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

which is the desired formaula.

(c) Since $r > 0$, then $\mu = \frac{1}{n}\log(1+r) > 0$. Since μ is positive, this subsequently implies that $e^{2\mu} > 1$. On the other hand, if we consider the function,

$$g(p) = 4p(1-p),$$

over the interval $p \in [0,1]$, then its extrema occur either at the endpoints, $p = 0, 1$, or at critical points, $p = 1/2$. The values of g at these candidate extrema are,

$$g(0) = g(1) = 0, \qquad g(1/2) = 1,$$

and hence we have,

$$0 \leq g(p) \leq 1, \qquad\qquad p \in [0, 1].$$

Therefore:

$$e^{2\mu} > 1 \geq g(p),$$

i.e.,

$$e^{2\mu} > 4p(1-p),$$

as desired, which implies that our formula for u corresponds to two distinct real values.

(d) If we choose the "minus" option for u, i.e.,

$$u = \frac{1}{2p}\left(e^{\mu} - \sqrt{e^{2\mu} - 4p(1-p)}\right),$$

then we seek to show the inequality,

$$\frac{1}{2p}\left(e^{\mu} - \sqrt{e^{2\mu} - 4p(1-p)}\right) < 1$$

To determine if this inequality is true, our first step is to rearrange it to the equivalent inequality,

$$e^{\mu} - 2p < \sqrt{e^{2\mu} - 4p(1-p)}. \tag{1}$$

Note that naively squaring both sides of the inequality is not a valid operation unless $|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)}$. (I.e., $-2 < 1$ does not imply that $(-2)^2 < 1^2$.). Hence, we must verify,

$$|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)} \iff (e^{\mu} - 2p)^2 < e^{2\mu} - 4p(1-p),$$

where we have used the fact that $e^{2\mu} - 4p(1-p) > 0$ from part (c). By expanding the quadratic term and simplifying, this last inequality is,

$$-4pe^{\mu} + 4p^2 < -4p + 4p^2 \iff e^{\mu} > 1,$$

which we have already established. Hence $|e^{\mu} - 2p| < \sqrt{e^{2\mu} - 4p(1-p)}$ is true, and therefore we may square both sides of (1) to obtain the following inequality that is equivalent to our desired one,

$$(e^{\mu} - 2p)^2 < e^{2\mu} - 4p(1-p),$$

but we have already shown that this is equivalent to $e^{\mu} > 1$, which is already established. Therefore, we indeed have $u < 1$ by choosing the "minus" option, and so the "plus" choice is the only valid choice.