Department of Mathematics, University of Utah

**Introduction to Mathematical Finance**
**MATH 5760/6890 – Section 001 – Fall 2024**
**Homework 9 Solutions**
**The Cox-Ross-Rubinstein Model**

**Due: Friday, Nov 8, 2024**

---

Submit your homework assignment on Canvas via Gradescope.

**1.)** Consider the *exact* normalizd per-period moment conditions for the CRR models on slide `D19-S05(a)` that relates the unknown CRR parameters $(p_n, u_n, d_n)$ to the assumed-known real-world drift and volatility $(\mu, \sigma)$ and also the period length $h_n = T/n$.

(a) Show that the *exact* solution to this system is given by,

$$p_n = \frac{1}{2}\left(1 + \sqrt{\frac{h_n \mu^2}{h_n \mu^2 + \sigma^2}}\right), \quad u_n = \exp\left(\sqrt{\sigma^2 h_n + h_n^2 \mu^2}\right), \quad d_n = \exp\left(-\sqrt{\sigma^2 h_n - h_n^2 \mu^2}\right)$$

(You may assume $\mu > 0$, although the results hold for general $\mu$ by using $|\mu|$ instead of $\mu$.)

(b) Use the above equations to show that the real-world CRR equations for $(p_n, u_n, d_n)$ are an approximation to this exact solution when $h_n \to 0$.

<u>**Solution**</u>:

(a) The exact moment conditions are,

$$d_n = \frac{1}{u_n}, \qquad \mu = \frac{2p_n - 1}{h_n} \log u_n, \qquad \sigma^2 = \frac{4p_n(1-p_n)}{h_n} (\log u_n)^2 .$$

We recall that for our tree models we assume $p_n \in (0, 1)$, and that $u_n > 1$. Note that $d_n$ can be computed once $u_n$ is known, so we focus on determining $(p_n, u_n)$ from the second and third equations. By squaring the second equation and dividing it by the third equation we obtain,

$$\mu^2 = \frac{(2p_n - 1)^2}{h_n^2} (\log u_n)^2 \quad \Longrightarrow \quad \frac{\mu^2}{\sigma^2} = \frac{(2p_n - 1)^2}{h_n 4p_n(1-p_n)},$$

where in order to perform the division we require $\log u_n \neq 0$, i.e., $u_n \neq 1$, which is satisfied by our assumptions. Therefore, if we define,

$$\alpha \coloneqq \frac{h_n \mu^2}{\sigma^2} > 0,$$

which is a known constant, then we have the equality,

$$\alpha = \frac{(2p_n - 1)^2}{4p_n(1-p_n)},$$

and the denominator cannot be zero since $p_n \neq 0$, $p_n \neq 1$. Rearranging the above equality, we have,

$$4p_n\alpha - 4p_n^2\alpha = 4p_n^2 - 4p_n + 1.$$

The roots $p_n$ of this quadratic equation are,

$$p_n = \frac{1}{2}\left(1 \pm \sqrt{\frac{\alpha}{\alpha+1}}\right).$$

Under the $\mu > 0$ assumptions, the "minus" is invalid; to see why, note that by our original moment conditions,

$$\mu = \frac{2p_n - 1}{h_n}\log u_n \qquad \Longrightarrow \qquad u_n = \exp\left(\frac{h\mu}{2p_n-1}\right).$$

And note that if we choose the "minus" root for $p_n$, then

$$2p_n - 1 = -\sqrt{\frac{\alpha}{\alpha+1}} \qquad \Longrightarrow \qquad u_n = \exp\left(-h\mu\sqrt{\frac{\alpha+1}{\alpha}}\right) < 1,$$

which violates our $u_n > 1$ assumption. However, choosing the "plus" option for $p_n$, i.e.,

$$\begin{aligned} p_n &= \frac{1}{2}\left(1 + \sqrt{\frac{\alpha}{\alpha+1}}\right) \\ &= \frac{1}{2}\left(1 + \sqrt{\frac{h_n\mu^2}{h_n\mu^2+\sigma^2}}\right), \end{aligned}$$

yields

$$\begin{aligned} u_n &= \exp\left(\frac{h_n\mu}{2p_n-1}\right) = \exp\left(h_n\mu\sqrt{\frac{\alpha+1}{\alpha}}\right) \\ &= \exp\left(\sqrt{\sigma^2 h_n + \mu^2 h_n^2}\right) > 0, \end{aligned}$$

and hence $(p_n, u_n)$ as derived above are our desired expressions. The formula for $d_n$ is obtained directly:

$$d_n = \frac{1}{u_n} \qquad \Rightarrow \qquad d_n = \exp\left(-\sigma\sqrt{h_n} - \mu h_n\right)$$

(b) A "direct" way to make the required argument is that when $h_n \to 0$, then

$$\frac{h_n\mu^2}{h_n\mu^2+\sigma^2} \to \frac{h_n\mu^2}{0+\sigma^2} = h_n\frac{\mu^2}{\sigma^2},$$

and therefore for small $h_n$:

$$p_n = \frac{1}{2}\left(1 + \sqrt{\frac{h_n\mu^2}{h_n\mu^2+\sigma^2}}\right) \to \frac{1}{2}\left(1+\sqrt{\frac{h_n\mu^2}{\sigma^2}}\right) = \frac{1}{2}\left(1+\sqrt{h_n}\frac{\mu}{\sigma}\right),$$

which matches the real-world CRR equation for $p_n$. For $u_n$, note that,

$$h_n \to 0 \qquad \Longrightarrow \qquad h_n^2 \ll h_n$$

and therefore,

$$h_n \to 0 \quad \Longrightarrow \quad \exp(\sqrt{\sigma^2 h_n + \mu^2 h_n^2}) \approx \exp(\sigma\sqrt{h_n}),$$

where the latter is true because $e^{\sqrt{ax+bx^2}} \approx e^{\sqrt{x}}$ for small $x > 0$, and any $a > 0$ and $b \in \mathbb{R}$. Hence,

$$h_n \to 0 \quad \Longrightarrow \quad u_n \approx \exp(\sigma\sqrt{h_n}),$$

which is the real-world CRR equation for $u_n$. A similar argument holds for $d_n$.

**2.)** Consider an $n = 100$-period real-world CRR model for a stock price with annual continuous-time drift of 15% and annual volatility 10%. Today's stock price is $S_0 = \$50$.

(a) Determine the parameters $(p_n, u_n, d_n)$ (with $n = 100$) corresponding to this CRR model with a terminal time of one year.

(b) Compute the expected stock price after one year.

(c) Compute the probability that the stock price will exceed its expected value.

(d) Using the same number of periods ($n = 100$) construct a real-world CRR model with terminal time of 6 months to compute the probability that the stock price is below or equal to today's price.

<u>**Solution**</u>:

(a) We are given a drift of $\mu = 0.15$ and a volatility of $\sigma = 0.1$. With a terminal time of one year $T = 1$, then $h_n = \frac{T}{n} = 0.01$. The real-world CRR equations yield

$$u_n = e^{\sigma\sqrt{h_n}} \approx 1.01, \qquad d_n = e^{-\sigma\sqrt{h_n}} \approx 0.99, \qquad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right) = 0.575$$

(b) Now that we have the triple $(p_n, u_n, d_n)$, we can utilize our known formulas for the expected value of an $n$-period binomial model (see, e.g., problem 2 on homework assignment 6). This yields:

$$\mathbb{E}S_n = \left(p_n u_n + (1 - p_n) d_n\right)^n.$$

Using $(p_n, u_n, d_n)$ as above with $n = 100$ and $S_0 = \$50$ yields,

$$\mathbb{E}S_n \approx 50 \times 1.168 \approx \$58.38.$$

(c) In order to determine when the stock price will exceed $\mathbb{E}S_n = \$58.38$, we need to determine the smallest value of $k$ such that,

$$50 u_n^k d_n^{n-k} > 58.38. \tag{1}$$

By direct computation, we have,

$$50 u_n^{57} d_n^{43} = 57.51$$
$$50 u_n^{58} d_n^{42} = 58.68.$$

Hence (1) is true for $k \geq 58$. Then the probability desired is the probability that a Binomial$(n, p_n)$ random variable has value 58 or greater, which is,

$$\sum_{k=58}^{100} \binom{100}{k} p_n^k (1 - p_n)^{100-k} \approx 0.502.$$

(d) With a new terminal time of $T = 0.5$, we must recompute the values $(p_n, u_n, d_n)$ from the real-world CRR equations with $h_n = 0.5/100 = 0.005$:

$$u_n = e^{\sigma\sqrt{h_n}} \approx 1.007, \quad d_n = e^{-\sigma\sqrt{h_n}} \approx 0.993, \quad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right) \approx 0.553$$

Because of the recombining condition of the CRR tree, we know that,

$$S_0 u_n^{50} d_n^{50} = S_0,$$

and hence we must have 50 or fewer upticks over 100 periods in order to end up at or below where we started. I.e., the probability of this happening is the probability that a Binomial$(100, p_n)$ random variable has value 50 or less, which is,

$$\sum_{k=0}^{50} \binom{100}{k} p_n^k (1-p_n)^{100-k} \approx 0.167.$$

**3.)** Choose your favorite stock, and collect daily historical data over a period of $[0, \widetilde{T}]$ (of at least one year in length, $\widetilde{T} \geq 1$). Use either the open or close price (do not use daily high or low prices). Use this data to compute (approximations to) the continuous-time drift $\mu$ and volatility $\sigma$. (Explain briefly the data that you used and what procedure you used to compute these values). Numerically simulate 10 trajectories of an $n = 252$-period corresponding real-world CRR model over a period of one year, $T = 1$, given the initial stock price $S_0 = S(0)$ from your data. Generate a plot of these realizations overlayed with the actual historical one-year data.

<u>**Solution**</u>:

(a) For no particularly good reason, I chose the stock GOOG (Google, now Alphabet, Inc.). I collected daily historical closing prices over the intervals $[0, \widetilde{T}]$ with $\widetilde{T} = 1$, with

$$t = 0 : \text{January 1, 2020}$$
$$t = 1 : \text{January 1, 2021 (no trading on this day)}.$$

which corresponded to 253 total data points. I.e., I have access to data,

$$S_0, S_1, \ldots, S_{252}$$

I assume the time instances $t_j$, for $j = 0, \ldots, 252$, are equally spaced, i.e., $h_n = 1/252$. The drift is the (normalized) empirical mean for sequential log-returns:

$$Y_j := \log\frac{S_j}{S_{j-1}}, \qquad \mu \approx \frac{1}{h_n}\mathbb{E}Y \approx \frac{1}{nh_n}\sum_{j=1}^{n} Y_j \approx 0.2477$$

and the variance if the normalized empirical variance for sequential log-returns:

$$\sigma \approx \sqrt{\frac{1}{h_n}\text{Var}Y} \approx \sqrt{\frac{1}{nh_n}\sum_{j=1}^{n}(Y_j - \mathbb{E}Y)^2} \approx 0.3838$$

(One may use the unbiased coefficient of $\frac{1}{n-1}$ instead of the maximum likelihood coefficient $\frac{1}{n}$ in the variance computation, but in practice for this exercise that

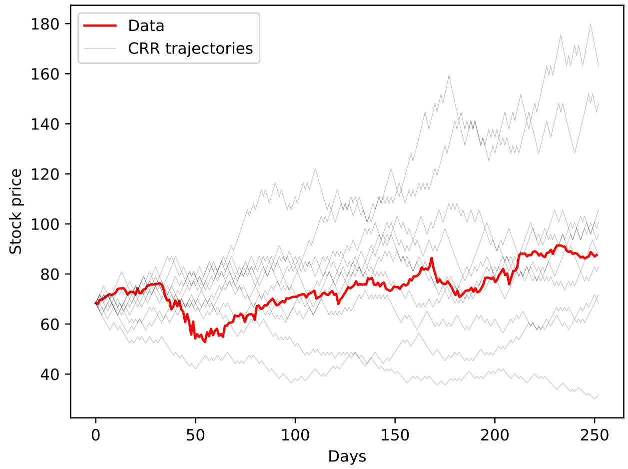


Figure 1: 10 simulations of a CRR model pegged to year-2020 daily historical data of the ticker GOOG.

would make negligible difference.) Note the rather high volatility for this example; part of this can be explained by the fact that the time period straddled the onset of the COVID-19 pandemic, during which stocks were unusually volatile due to investor uncertainty about the future. With these values, we can immediately use the real-world CRR formulas to compute:

$$u_n = e^{\sigma\sqrt{h_n}} \approx 1.0245, \quad d_n = e^{-\sigma\sqrt{h_n}} \approx 0.9761, \quad p_n = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h_n}\right) \approx 0.5203.$$

A simulation with 10 trajectories of this CRR model, with the real stock data shown, is plotted in figure 1.