Department of Mathematics, University of Utah

Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2024
Homework 9 Solutions
The Cox-Ross-Rubinstein Model

Due: Friday, Nov 8, 2024

Submit your homework assignment on Canvas via Gradescope.

1.) Consider the *exact* normalizd per-period moment conditions for the CRR models on slide `D19-S05(a)` that relates the unknown CRR parameters (p_n, u_n, d_n) to the assumed-known real-world drift and volatility (μ, σ) and also the period length $h_n = T/n$.

(a) Show that the *exact* solution to this system is given by,

$$p_n = \frac{1}{2}\left(1 + \sqrt{\frac{h_n\mu^2}{h_n\mu^2 + \sigma^2}}\right), \quad u_n = \exp\left(\sigma\sqrt{h_n} + h_n\mu\right), \quad d_n = \exp\left(-\sigma\sqrt{h_n} - h_n\mu\right)$$

(You may assume $\mu > 0$, although the results hold for general μ by using $|\mu|$ instead of μ.)

(b) Use the above equations to show that the real-world CRR equations for (p_n, u_n, d_n) are an approximation to this exact solution when $h_n \to 0$.

2.) Consider an $n = 100$-period real-world CRR model for a stock price with annual continuous-time drift of 15% and annual volatility 10%. Today's stock price is $S_0 = \$50$.

(a) Determine the parameters (p_n, u_n, d_n) (with $n = 100$) corresponding to this CRR model with a terminal time of one year.

(b) Compute the expected stock price after one year.

(c) Compute the probability that the stock price will exceed its expected value.

(d) Using the same number of periods ($n = 100$) construct a real-world CRR model with terminal time of 6 months to compute the probability that the stock price is below or equal to today's price.

3.) Choose your favorite stock, and collect daily historical data over a period of $[0, \widetilde{T}]$ (of at least one year in length, $\widetilde{T} \geq 1$). Use either the open or close price (do not use daily high or low prices). Use this data to compute (approximations to) the continuous-time drift μ and volatility σ. (Explain briefly the data that you used and what procedure you used to compute these values). Numerically simulate 10 trajectories of an $n = 252$-period corresponding real-world CRR model over a period of one year, $T = 1$, given the initial stock price $S_0 = S(0)$ from your data. Generate a plot of these realizations overlayed with the actual historical one-year data.