Department of Mathematics, University of Utah
Introduction to Mathematical Finance
MATH 5760/6890 – Section 001 – Fall 2024
Project 2
Correlated geometric Brownian motion

Due: Thursday, Dec 5, 2024

Submit your assignment on Canvas via Gradescope.
In this project, you will construct a stochastic model to predict the future value of asset prices using real data. Complete the following steps, submitting a report-like description of how you completed these steps along with a presentation of the data or simulation results. You need not submit the code used to generate your results, but you should explicitly document how you collected data and what steps your performed to analyze the data resulting in simulations.

a. Identify N stocks/tickers that you wish to build a stochastic model for. Choose at least 6 stocks, i.e., $N \geq 6$. These can be stocks for publicly traded companies, or can themselves be hedge/mutual funds; however, they should be risky securities, i.e., they should have the potential to decrease in value. Collect daily stock data over some time interval $T \geq 1$ (in years).

b. Research and compile drift and volatility statistics for the normalized log returns of the above securities. Note that this is not the same as first- and second-order statistics for the actual security prices. In particular, recalling the procedure in homework 9:

- S_j is the asset price at time t_j, with $t_j = jh_n$ and $h_n = T/n$. The random variable $Y_j = \log S_j/S_{j-1}$ is the interperiod log-return.
- The approximation to the continuous-time drift and squared volatility of this asset is given by, respectively,

$$\mu \approx \frac{1}{h_n}\mathbb{E}Y \approx \frac{1}{nh_n}\sum_{j=1}^{n} Y_j, \qquad \sigma_S^2 \approx \frac{1}{h_n}\mathrm{Var}\, Y \approx \frac{1}{nh_n}\sum_{j=1}^{n}(Y_j - \mathbb{E}Y)^2,$$

  where $\mathbb{E}Y$ in the formula for σ_S^2 is approximated by $h_n\mu$.
- We introduce the covolatility between assets S_j and R_j, which is approximated by,

$$\sigma_{S,R} \approx \frac{1}{h_n}\mathrm{Cov}\,(Y, Z) \approx \frac{1}{nh_n}\sum_{j=1}^{n}(Y_j - \mathbb{E}Y)(Z_j - \mathbb{E}Z),$$

  where $Y_j = \log S_j/S_{j-1}$ as before, and $Z_j = \log R_j/R_{j-1}$. In particular, $\sigma_{S,S} = \sigma_S^2$.

c. Construct an N-dimensional system of SDEs for correlated geometric Brownian motion. I.e., let $\boldsymbol{S} = \boldsymbol{S}(t) \in R^N$ denote the size-N vector of asset prices at time t. This should evolve according to the SDE,

$$\mathrm{d}\boldsymbol{S} = \left[\boldsymbol{\mu} \circ \boldsymbol{S} + \frac{1}{2}\boldsymbol{\sigma}^2 \circ \boldsymbol{S}\right] \mathrm{d}t + \boldsymbol{S} \circ (\sqrt{\boldsymbol{\Sigma}}\, \mathrm{d}\boldsymbol{B}), \tag{1}$$

where $\circ$ represents componentwise/elementwise multiplication between vectors. We also have:

- $\boldsymbol{\mu} \in \mathbb{R}^N$ has entries $\mu_1, \ldots, \mu_N$, which are the continuous-time drifts for the securities $S_1, \ldots, S_N$ computed in the previous part.
- $\boldsymbol{\sigma}^2 \in \mathbb{R}^N$ has entries $\sigma_1^2, \ldots, \sigma_N^2$, which are the continuous-time squared volatilities for the securities $S_1, \ldots, S_N$ computed in the previous part.
- $\boldsymbol{B} = \boldsymbol{B}(t) \in \mathbb{R}^N$ is a vector containing N independent standard Brownian motions. (I.e., one generates increments of each entry via independent Gaussian random variables.)
- $\boldsymbol{\Sigma} \in \mathbb{R}^{N \times N}$ is the positive semi-definite covolatility matrix containing the co-/volatilities computed in the previous part, i.e.,:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \sigma_{N,1} & \cdots & & \sigma_N^2 \end{pmatrix},$$

  where σ_j^2 is the squared volatility of security j, and $\sigma_{j,k} = \sigma_{k,j}$ is the covolatility of securities j and k. In addition, $\boldsymbol{\sigma} = \text{diag}(\boldsymbol{\Sigma})$.
- The matrix $\sqrt{\boldsymbol{\Sigma}} \in \mathbb{R}^{N \times N}$ is a(ny) matrix square root of $\boldsymbol{\Sigma}$. E.g., one option is through an eigenvalue decomposition:

$$\boldsymbol{\Sigma} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^T \quad \Longrightarrow \quad \sqrt{\boldsymbol{\Sigma}} = \boldsymbol{V}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{V}^T.$$

  Above, $\boldsymbol{\Lambda}$ is a diagonal matrix containing the (non-negative) eigenvalues of $\boldsymbol{\Sigma}$, and $\boldsymbol{V}$ is a vector whose columns contain the eigenvectors of $\boldsymbol{\Sigma}$. (Another option for $\sqrt{\boldsymbol{\Sigma}}$ is to use the lower-triangular Cholesky factor of $\boldsymbol{\Sigma}$.)

d. Simulate 10 trajectories of the SDE system (1) starting from time $t = 0$ up to time $t = T$, overlayed with the original data. (Use the actual security prices at $t = 0$ as the initial condition.) Create N plots, where plot i contains the plot of the 10 trajectories of ith component of $\boldsymbol{S}$, plus the data you collected for security i. (Note that this is not $10N$ simulations; it is 10 simulations, where each simulation has N components.) You may simulate the SDE (1) using the Euler-Maruyama method which takes the form,

$$\boldsymbol{S}_{j+1} = \boldsymbol{S}_j + \Delta t \left(\left(\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\sigma}^2 \right) \circ \boldsymbol{S}_j \right) + \sqrt{\Delta t}\boldsymbol{S}_j \circ \left(\sqrt{\boldsymbol{\Sigma}}\boldsymbol{Z} \right), \qquad \boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}),$$

where $\boldsymbol{S}_j$ denotes the numerical approximation to $\boldsymbol{S}(j\Delta t)$. Take Δt to correspond to at most an increment of 1 trading day. (Shorter is fine.) The initial condition $\boldsymbol{S}_0 = \boldsymbol{S}(0)$ should be given by the initial time stock prices collected in part a.