

Math 5760/6890: Introduction to Mathematical Finance

See Petters and Dong 2016, Sections 1.1-1.3

Akil Narayan¹

¹Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

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Mathematical/quantitative finance is the science of determining appropriate prices of financial products.

- Personal banking
 - ▶ checking, savings, money market accounts
 - ▶ certificates of deposit
 - ▶ traveler's checks/money orders
 - ▶ mortgages (home/property loans)
 - ▶ auto or personal loans
 - ▶ debit/credit cards
- Investment banking
 - ▶ stocks, bonds, mutual funds
 - ▶ options
 - ▶ derivative securities

This science is exercised worldwide by banks, traders, governments, companies, researchers, etc.

Markets are exchanges between those seeking to invest, save, or attract capital.

Typically these savers, investors, and lenders are

- you and me (consumers)
- governments
- private companies

These players seek to buy and sell financial products.

Banks are intermediaries between these players in the market.

There are different types of banks, but in this class we typically won't worry about this distinction.

Loosely speaking, banks pool saved money and use it to make investments and loans.

Consider the following scenario:

- I deposit \$100 in a checking account.
- Using my funds, the bank lends out \$50 to person A for a period of time.¹
- I ask for all my money back before person A can repay the loan.

The bank is short on funds to accomplish this repayment to me.

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The ratio of liabilities (debt) to actual immediately available equity is the **leverage ratio**.

Other side of the same coin: the percentage of a deposit that the bank reserves for equity safekeeping is the **reserve ratio**.

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The downside: In the example above, the bank has a reserve ratio of 50%, and a leverage ratio of 2. (Typically written as “2:1”.)

The upside: In terms of *total* credit+debt generated in this example, my initial \$100 generated \$150 in financial products.

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Leverage to the extreme

D02-S05(a)

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Leverage to the extreme

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- Using my funds, the bank lends out \$50 to person A for a period of time.
- Person A buys a thingamabob from person B for \$50.
- Person B deposits the \$50 into a bank (perhaps a different one)
- The bank reserves \$25, and loans out \$25 to person C.
- Person C buys

What is the total credit/debt created by continuing this process indefinitely assuming the same reserve ratio?

- \$100: amount I have credited the bank (the bank's debt)
- \$50: amount the bank has credited person A (person A's debt)
- \$25: amount the second bank has credited person C (person C's debt)
- ...

In total:

$$100 + 50 + 25 + \dots = 100 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{100}{1 - \frac{1}{2}} = 200$$

Hence, a uniform leverage ratio of $R:1$ (reserve ratio of $\frac{100}{R}\%$) can generate up to $100R$ in financial products.

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High leverage corresponds to high risk. (If one creditor calls in a debt, it's a falling dominos situation, with the equity shortfall directly related to the leverage ratio.)

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Many countries have non-zero reserve requirements, and reserve requirements are only one regulatory tool to curb financial engineering running amok.

An important factor in financial engineering is **interest**, which is the price of credit, that is of holding money or capital (typically someone else's money).

Whenever some entity (the lender) loans money or assets to another entity (the lende), the lender typically charges interest.

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There are different types of interest rates, and different ways to compute interest – this topic will be our first major consideration in this class.

The central goal in finance:

Given enough information about financial products – types of products, prices on different markets, transaction fees, interest, etc. – craft a policy, i.e., a simultaneous/sequential series of actions that increases the sum total of your assets.

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A somewhat nuanced strategy for achieving a profit: **arbitrage**.

This is the practice of identifying and implementing a profit-making policy by leveraging a difference in price of an asset. Arbitrage is a “sure-thing” profit strategy.

For example:

- You find the course textbook available from an online bookseller for \$100, all expenses included.
- The campus bookstore is offering to buy the course textbook for \$200.

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An arbitrage typically exists when there is some kind of inequity in the market: e.g., you have information that others don't have, or you have fewer expenses than others have.

There is typically *no arbitrage* in markets where everyone shares the same information, expenses, and opportunities.

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The typical way to model uncertainty in markets is to model outcomes using *probability*. (This will be the focus for the last ~half of the semester.)

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For future reference: in financial math, the more formal definition of an arbitrage is that it is a strategy for which both of the following are true:

- there is zero probability of losing money
- there is positive probability of gaining money

I.e., you can't lose, but you may win.

There are several types of financial securities.

Example (Stocks)

A **stock** is a partial ownership contract; ownership of stock entitles the bearer to a fractional ownership of an entity (typically a company).

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There are several types of shares of stock, which offer different benefits

- common stock
- preferred stock
- convertible stock
- Class A, B, C shares

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A **bond** is a (type of) loan: the issuer (debtor) is “obliged” to pay back the holder (creditor) a fixed amount after a fixed period of time.

Bond issuers may be companies or entities, but holders of a bond do not enjoy ownership benefits – they are simply lenders.

Office Hours : WFB 4666

- Wed 10:30 - 11:15 am

- Thurs 10:15 - 11:00 am

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The bond yield is derived from the above, typically referring to rate of return of the bond.

Bonds are frequently sold below par and reach par value at maturity.

Some bonds have fixed rates, some have variable (say inflation-adjusted) rates.

“Obliged” means that the issuer is required to pay back the holder with near-ultimate priority. (E.g., bondholders are typically paid before stock holders are paid.)

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Example (Forward contracts)

A **forward contract** is an agreement between parties to conduct a transaction at a future time with a specified price.

The buyer in this contract is hoping that the asset will increase in value beyond the agreed upon price (a “long position”).

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The value of a forward contract to either the buyer or the seller is tied to, or derived from, the performance of an underlying asset.

Hence, this is a type of **derivative** financial instrument: i.e., the *contract itself*, not the underlying asset in the contract transaction, is a financial product that can be bought and sold.

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An **option** is the privilege (but not the obligation) to exercise a transaction at a specified (“strike”) price on or before a certain date.

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Two main types of options are:

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A *rational* call option holder typically only exercises the option if the strike price is below market value.

A *rational* put option holder typically only exercises the option if the strike price is above market value.

Options have styles; the “vanilla” ones are American and European options.

This course studies financial security assets, typically with the goal of understanding

- valuation, either at present or in the future
- risk
- strategies for mitigating risk and maximizing profit



Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.