Math 5760/6890: Introduction to Mathematical Finance

See Petters and Dong 2016, Sections 1.1-1.3

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In either case, any investment opportunity for the investor should be evaluated by weighing the investment payoff against the alternative investment payoff.

The **present value** of an investment is today's value of future investment payoffs, factoring in an interest "discount".

Example

Suppose you are offered to invest in a company (e.g., purchase a stock share), and from your investment you are promised the following payments due to growth of the company:

- End of year 1: \$50
- End of year 2: \$75
- End of year 3: \$100

In order to determine today's value of this investment opportunity, suppose an alternative opportunity ensures an annual growth rate $r = 10\%$.

\$50 in 1 year is $\frac{50}{(1+0.1)} = \frac{50}{1.1}$ dollars today

\$75 in 2 year is $\frac{75}{(1+0.1)^2}$ dollars today (compounded annually)

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- End of year 2: \$75
- End of year 3: \$100

In order to determine today's value of this investment opportunity, suppose an alternative opportunity ensures an annual growth rate $r = 10\%$.

Then the present value of this opportunity is,

$$PV = \frac{50}{1+r} + \frac{75}{(1+r)^2} + \frac{100}{(1+r)^3} \approx \$182.57$$

(r = 0.1)

(If the alternative is due to interest, compounding should be taken into consideration.)

Terminology: the rate r (of an alternative investment) is called a "required return rate".

Net present value

One can compute the present value of an investment opportunity if fixed payoffs are assured and an alternative opportunity rate is identified.

To decide if this is a good opportunity, we need information about the price/capital required in the initial payment.

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Example

The opportunity in the previous example had a present value of \$182.57. If the price of this investment or initial capital requirement is \$150, then the **net present value** of this opportunity is defined as,

$$NPV = \$182.57 - \$150.$$

If $NPV \geqslant 0$, this opportunity is attractive. It generally is not attractive if $NPV < 0$.

Internal return rate

The return rate from an investment is computed using a similar idea.

Example

The annual return rate for the investment in the previous examples is the rate r at which the r-discounted payoffs match the initial price:

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The annual return rate for the investment in the previous examples is the rate r at which the r-discounted payoffs match the initial price:

$$150 = \frac{50}{1+r} + \frac{75}{(1+r)^2} + \frac{100}{(1+r)^3}$$

We can, in principle, solve for r:

$$x = \frac{1}{1+r} \implies 100x^3 + 75x^2 + 50x - 150 = 0 \implies x = 0.82909...$$

$$\implies r \approx 0.206.$$

Hence, this opportunity corresponds to an approximately 20% return rate.
(So that there is a positive NPV for an alternative 10% return rate is unsurprising.)

Of course, the PV of this opportunity at the *internal* return rate should equal $150.

Given an alternative investment with required return rate r, and an opportunity with a net present value NPV and internal return rate r_I, we have two ways to determine whether the opportunity is attractive:

- If $NPV \geqslant 0$, then the opportunity is attractive
- If $r_I \geqslant r$, then the opportunity is attractive

Are these criteria the same?

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Theorem

Assume every periodic payment is strictly positive. Suppose there is some positive number r_I corresponding to the internal return rate. Then r_I is unique, and:

- $NPV > 0 \iff r_I > r$
- $NPV = 0 \iff r_I = r$
- $NPV < 0 \iff r_I < r$

(So, yes, these are the same thing.)

Application: annuities

An **annuity** is an agreement for a series of payments made at a regular period for a fixed term.

- loans and debt with interest (e.g., mortgages, car loans, credit card balances)
- pension plan payments
- social security

Note that whether we *receive* payment or *make* payments, the problem of valuation of these products identical.

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You receive a monthly payment of $100 monthly over a term of 3 years.

First let's determine the *future* value of this annuity, i.e., the worth of this annuity after 3 years.

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We'll assume the ability to invest money with an interest rate r, compounded monthly.

This translates into:

- The \$100 after the end of month 1 is invested for 35 months with future value $100\left(1+\frac{r}{12}\right)^{35}$
- The \$100 after the end of month 2 is invested for 34 months with future value $100\left(1+\frac{r}{12}\right)^{34}$
- The \$100 after the end of month 3 is invested for 33 months with future value $100\left(1+\frac{r}{12}\right)^{33}$
- $\vdots$

Annuities: future value

Hence, the future value of this annuity is:

$$
\begin{aligned}
FV &= 100\,(1+\frac{r}{12})^{35} + 100\,(1+\frac{r}{12})^{34} + \cdots + 100 \\
&= 100 \sum_{j=0}^{35} \left(1+\frac{r}{12}\right)^{j} \\
&= 100\,\frac{\left(1+\frac{r}{12}\right)^{36} - 1}{1+\frac{r}{12}-1} \\
&= \frac{1200}{r}\left[\left(1+\frac{r}{12}\right)^{36} - 1\right]
\end{aligned}
$$

$$\left(\sum_{j=0}^{n} x^j = \frac{x^{n+1}-1}{x-1} \right)$$

Hence, the future value of this annuity is:

$$\begin{aligned} FV &= 100\,(1+\frac{r}{12})^{35} + 100\,(1+\frac{r}{12})^{34} + \cdots + 100 \\ &= 100 \sum_{j=0}^{35} \left(1+\frac{r}{12}\right)^{j} \\ &= 100 \frac{\left(1+\frac{r}{12}\right)^{36} - 1}{1+\frac{r}{12}-1} \\ &= \frac{1200}{r}\left[\left(1+\frac{r}{12}\right)^{36} - 1\right] \end{aligned}$$

With some calculus exercises one can confirm expected behavior of this future value on r:

- FV increases as r increases (for more than 1 period)
- FV accelerates as r increases (for more than 2 periods).

Annuities: present value

In many cases, today's value (the present value) of the annuity is more useful than the future value.

The present value of this annuity is determined by its future value discounted by interest rate r compounded monthly:

- The \$100 payment after the end of month 1 has PV $\frac{100}{1+\frac{r}{12}}$.
- The \$100 payment after the end of month 2 has PV $\frac{100}{(1+\frac{r}{12})^2}$.
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For 36 periods, we then have the following present value:

$$PV = \underbrace{\frac{100}{1+\frac{r}{12}}}_{\text{period 1}} + \underbrace{\frac{100}{\left(1+\frac{r}{12}\right)^2}}_{\text{period 2}} + \underbrace{\frac{100}{\left(1+\frac{r}{12}\right)^3}}_{\text{period 3}} + \cdots$$

$$= 100 \sum_{j=1}^{36} \frac{1}{\left(1+\frac{r}{12}\right)^j}$$

$$= 100 \left(\frac{1 - \frac{1}{(1+\frac{r}{12})^{37}}}{1 - \frac{1}{1+\frac{r}{12}}} - 1 \right) = \frac{1200}{r} \left(1 - \frac{1}{\left(1+\frac{r}{12}\right)^{36}} \right)$$

Using the same computations as in the previous slides, we can make abstract statements about annuity valuations:

Assume a simple annuity is paid over a total of n periods with a periodic payment P. Assume an annual interest rate r compounded over k periods per year, coinciding with the annuity payments.
(Hence this annuity extends for n/k years.)

Previous examples $P = 100$

$n = 36$

$k = 12$

$r = r$

Annuities, summarized

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Then:

- The future value of the annuity (at the end of the annuity term) is

$$FV = \frac{Pk}{r}\left[\left(1+\frac{r}{k}\right)^n - 1\right].$$

- The present value of the annuity (today) is

$$PV = \frac{Pk}{r}\left(1 - \frac{1}{\left(1+\frac{r}{k}\right)^n}\right).$$

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The annuity model is not restricted to valuation of standard notions of annuities:

A lender can determine the principal for a loan by assuming the lendee is paying an annuity periodically with a given interest rate.

Example 2.13. (Paying Off Debt) Suppose that you borrow \$100,000 at an annual interest rate of 6% with monthly compounding. For an ordinary annuity based on this compounding, what is your minimum payment per month to pay off the loan in 10 years?

PV: \$100,000

$r = 6\%$

monthly payment of P

120 periods (10 yrs)

PV of pymt at end of month 1: $\frac{P}{\left(1+\frac{r}{12}\right)}$

" " " " " " 2: $\frac{P}{\left(1+\frac{r}{12}\right)^2}$

PV of all pymts: $\sum_{j=1}^{120} \frac{P}{\left(1+\frac{0.06}{12}\right)^j}$

$$PV = P \sum_{j=1}^{120} \rho^j, \qquad \rho = \frac{1}{1 + \frac{0.06}{12}}$$

$$= P\left(\sum_{j=0}^{120} \rho^j - 1\right) = P\left(\frac{1-\rho^{121}}{1-\rho} - 1\right) = P\left(\frac{1-\rho^{121} - 1 + \rho}{1-\rho}\right)$$

$$= \frac{P\rho}{1-\rho}\left(\rho^{120} - 1\right)$$

$PV = 100{,}000$

$$\Rightarrow P = \frac{100{,}000}{\rho}(1-\rho)\frac{1}{\rho^{120}-1} = \text{some number.}$$

Another example

Consider the setup of the previous example: you borrow \$100,000 with an annual interest rate of 6% compounded monthly, and you intend to pay \$1,000 per month toward the loan. How many years will it take for you to repay the loan?

PV ✓

$\mathbb{P}$ ✓

r ✓

n ?

From before: $PV = \mathbb{P}\frac{\rho}{1-\rho}(\rho^n - 1)$

$\rho = \frac{1}{1 + \frac{0.06}{12}}$

↓ solve for n

\# years is $n/12$

Present value is also useful for valuation of stocks and bonds:

The **dividend discount model** (DDM) is the assumption that the value of a stock (today) is given by the sum of (a) the present value of the stock's future price and (b) the present value of the dividends the company pays out.

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Suppose you purchase stock in a company that today paid a dividend of D_0 (say per share). For modeling purposes, we'll assume:

- You hold the stock for n periods, at which time you liquidate your shares.
- After n periods, the stock share will be priced at P_n.
- The company will pay a dividend every period that grows with *per-period* rate r_D.
- The *per-period* internal return rate r_I of the company is known.
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- If the company had *not* paid dividends, then the money would have grown at the company's internal return rate.

Then according to the DDM, the present value of a stock share is:

$$PV = \underbrace{\frac{P_n}{(1+r_I)^n}}_{\text{PV of share price}} + \sum_{j=1}^{n} \underbrace{\frac{D_0(1+r_D)^j}{(1+r_I)^j}}_{\text{PV of period-}j\text{ dividend}}$$

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.