Math 5760/6890: Introduction to Mathematical Finance

See Petters and Dong 2016, Sections 7.2-7.5

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A review of some concepts:

- Taking a *long position* in a security entails purchasing it with the hope that it will increase in value.
- Taking a *short position* in a security entails selling it with the hope that it will increase in value. *decrease*
  Short positions often require borrowing the security when you don't have it.

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However, these terms are used to generically describe any security, including a derivative.

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Market sentiment is used to describe a general view of a security.

- A *bullish sentiment* of a security indicates a general expectation of increase in value.
- A *bearish sentiment* of a security indicates a general expectation of decrease in value.

Taking a long position in a security is a bullish bet on that security.
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Forward contracts are simple. They are agreements

- typically between two parties, the *buyer* and *seller*
- that specify a transaction to happen at a specified time and quantity
  - the buyer agrees to purchase an *underlier*, i.e., the asset, from the seller
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  ‣ the buyer agrees to purchase an *underlier*, i.e., the asset, from the seller
  ‣ the *delivery date* or *expiry* is the future transaction date
- with a specified *forward price* and *contract size* of the underlier
- in which ~~typically~~ *often* no payment is made upon initiation.
- that are *not* standardized.
  ‣ payment details can be arbitrary
  ‣ e.g., a down payment, involving payment of a portion of the forward price, may be involved

Forward contracts are simple enough to consider, and are quite popular:

- businesses often engage in vanilla forward contracts to ensure timely and predictable availability of products to customers
- individuals often enter into subscription-type forward contracts for commodities
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Forward contracts are not generally regulated, and are frequently (not always) off-the-market agreements between two entities.

The *value* of a forward contract depends on <u>time</u>.
That is, the underlier can change value despite an initially agreed upon forward price.

- Let $t = 0$ be the time at which a forward contract is initiated
- Let $t = T$ be the delivery date
- Let $F_T(0)$ be the forward price: the per-unit price agreed upon at time 0 that comes due at time T.
- Let $S(t)$ denote the per-unit underlier price
- The buyer's per-unit "profit", or the seller's loss, is $S(T) - F_T(0)$.
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If we model uncertainty about the future as randomness in S, then $S(T) - F_T(0)$ is random.

What's more: this randomness depends on the current value of t.

I.e., conditioned on the current time t, the random processes,

$$S(T|t = 0), \qquad S(T|t = t > 0),$$

have different distributions. Hence, the forward contract's long position payoff,

$$S(T|t) - F_T(0),$$

is a t-dependent random process.

Futures ("future contracts")

    – are "standardized" forward contracts

    – are regulated and sold on markets

    – are therefore useful derivatives or as ingredients in derivatives

    – typically involve a market exchange as a party

    – have essentially the same terminology as a forward contract

    – require *daily settlements* with *margins*

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Swaps ("swap contracts")

- are an exchange of securities that are typically of cash-flow type (most commonly interest payments)
- typically involve larger institutions and corporations, not individuals

A plain vanilla swap is an exchange of fixed-interest income ("fixed leg") for floating interest income ("floating leg").

Options are more complicated versions of standardized forward contracts.

Recall:

- The buyer pays a premium at the initiation of the option contract
- The buyer purchases the right to conduct a transaction at a specified *strike* price K in the future. (This is the options analog of the "forward price".)
  - ▸ (European-style) In a simple case, this option may be exercised only at expiry, but not before.
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The American options are also much harder to price (fair premium determination).

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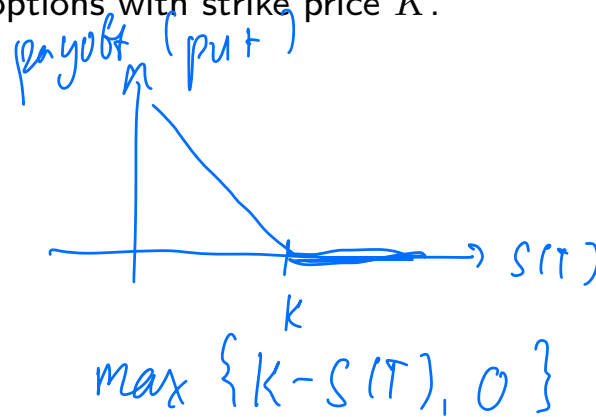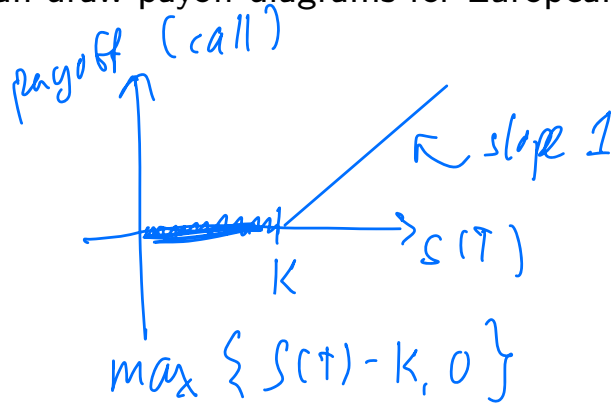
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Note again, that in practice the payoff $\max\{S(T) - K, 0\}$ is unknown (say, random) and depends on the current time t!

Visualizing options payoffs can be more intuitive:

Recall that, for European options, a payoff diagram plots the per-unit payoff versus the spot price $S(T)$. E.g., we can draw payoff diagrams for European call and put options with strike price K.

payoff (call)

slope 1

$S(T)$

K

$\max\{S(t) - K, 0\}$

payoff (put)
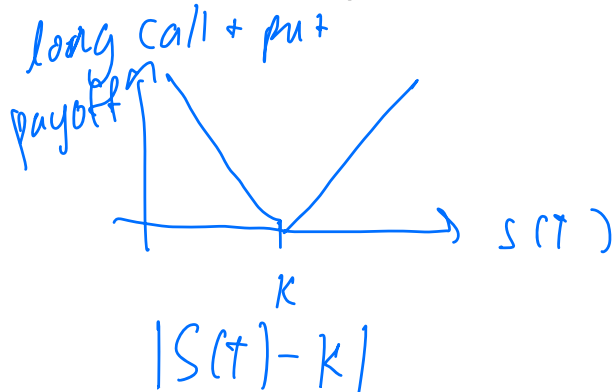
$S(T)$

K

$\max\{K - S(T), 0\}$

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To more intuitively visualize such portfolio constructions, one typically draws a *profit* diagram, which vertically shifts the payoff diagram by the (net) premium (usually the future value thereof).

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.