Math 5760/6890: Introduction to Mathematical Finance

2-Security Portfolios

See Petters and Dong 2016, Sections 3.1-3.2

Akil Narayan[1]

[1]Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

Fall 2024

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- the future behavior of assets is uncertain or random

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Because of the reliance on first- and second-order statistics, Markowitz (or "Modern") portfolio theory is an example of *mean-variance* analysis.

Investors in the one-period model

We will consider the one-period model:

- Today at time $t = 0$ all assets have known, deterministic prices
- "Tomorrow" at time $t = T > 0$, the asset prices become random variables
- No actions are possible between $t = 0$ and $t = T$ (no transactions, adjustment of exposure, etc.)

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We will also assume that investors have an internal understanding of their own risk tolerance as a function of reward.

Our job is to engineer a portfolio achieving, e.g., optimal risk for a fixed given reward.

Efficient portfolios

We have discussed enough to articulate a slightly more quantitative description of our goal.

Let $V = V(t)$ be the (total) value of the portfolio, so that

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Definition

A portfolio is an **efficient** portfolio if one *cannot*

- decrease risk without decreasing reward
- increase reward without increasing risk

Notation and setup

We have already discussed seen notation that parameterizes portfolios. Let's review:

- A portfolio is comprised of N securities, $N \geqslant 2$.
- The price per share/unit of security j is $S_j(t)$.
- The number of shares (fractional ok) we invest in security j is n_j.
- We are given an initial amount of capital, $V(0) > 0$, which is the initial value of the portfolio.
- Hence, the relative amount of money we invest in security j is given by the weight $w_j = \frac{n_j S_j(0)}{V(0)}$.

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We'll use vector versions of all these:

- $\boldsymbol{S}(t)$: the asset/security prices
- $\boldsymbol{n}$: the trading strategy
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$R = R_P$

Our eventual goal will be to consider the return rate R_P of the portfolio over the time interval $[0, T]$:

$$V(t) = \langle \boldsymbol{S}(t), \boldsymbol{n} \rangle, \qquad R_P(t) := \frac{V(t) - V(0)}{V(0)} = \left\langle \frac{\boldsymbol{S}(t)}{V(0)}, \boldsymbol{n} \right\rangle - 1.$$

Portfolio weights

In general we allow *short selling*, which corresponds to allowing negative weights/shares in the portfolio.

Example

Suppose we have a 3-security portfolio with initial value $\$1000$, where my 3 securities are:

- Security 1: Microsoft stock, $S_1(0) = \$100$
- Security 2: Coca-cola stock, $S_2(0) = \$50$
- Security 3: Ford Motor Company stock, $S_3(0) = \$150$

Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

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Using $w_1 V(0) = \$500$ in capital, this portfolio holds $n_1 = \frac{w_1 V(0)}{S_1(0)} = 5$ shares of Microsoft.

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Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

This portfolio holds $n_2 = \frac{w_2 V(0)}{S_2(0)} = -6$ shares of Coca-cola.

This means that we *borrow* 6 shares of Coca-cola from person A (like a "loan").

Putatively, we sell these shares back to the market (immediately) to generate $6S_2(0) = \$300$ in capital.

At time $t = T$ in the future, we need to repurchase + return these shares to person A: to expect a profit we hope that $S_2(T) < S_2(0)$.

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Suppose I form a portfolio with weights $\boldsymbol{w} = (0.5, -0.3, 0.8)^T$.

This portfolio holds $\frac{w_3 V(0)}{S_3(0)} = \frac{16}{3}$ shares of Ford using $\$800$ in capital.

This capital is comprised of $500 leftover from initial capital plus $300 in capital raised by shorting security 2.

This portfolio is *leveraged*. (We have an outstanding debt in Coca-cola stock required to form the portfolio.)

Return rates

We are eventually interested in the return rate $R_P(T)$ of the portfolio. It is notationally useful to rewrite things in terms of return rates for each individual security:

$$R_i(t) := \frac{S_i(t) - S_i(0)}{S_i(0)}, \qquad \boldsymbol{R}(t) = \left(R_1(t), R_2(t), \ldots, R_N(t)\right)^T.$$

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This definition implies that the full portfolio's return can be written as:

$$\begin{aligned}
R_P(T) = \frac{V(T) - V(0)}{V(0)} &= \frac{\langle \boldsymbol{S}(T), \boldsymbol{n}\rangle - V(0)}{V(0)} \\
&= \frac{\langle \boldsymbol{S}(T), \boldsymbol{n}\rangle - \langle \boldsymbol{S}(0), \boldsymbol{n}\rangle}{V(0)} \\
&= \frac{\langle \boldsymbol{S}(T) - \boldsymbol{S}(0), \boldsymbol{n}\rangle}{V(0)} \\
&= \sum_{j=1}^N n_i \frac{S_i(T) - S_i(0)}{V(0)} \\
&= \sum_{j=1}^N \frac{n_i S_i(0)}{V(0)} \frac{S_i(T) - S_i(0)}{S_i(0)} \\
&= \sum_{j=1}^N w_i R_i(T) = \langle \boldsymbol{R}(T), \boldsymbol{w}\rangle.
\end{aligned}$$

Return rates, II

We are interested in the return rate $R_P(T)$ of the portfolio.

$$R_P = \langle \boldsymbol{R}, \boldsymbol{w} \rangle, \qquad\qquad R_i(t) := \frac{S_i(t) - S_i(0)}{S_i(0)}.$$

Return rates are dimensionless quantities, and can be represented as a percentage.

Thus, it's easy to compare a portfolio's return rate to, e.g., an alternative risk-free interest rate.

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Under the Markowitz model, we focus on the expected return rate and variance (squared risk) of the return rate:

$$\text{Expected return rate} = \text{“}\mu_P\text{”} = \mathbb{E} R_P,$$
$$\text{Risk} = \text{“}\sigma_P\text{”} = \sqrt{\text{Var}\, R_P}$$

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A risk-*optimal* Markowitz portfolio will have the smallest σ_P for a fixed μ_P.
(This does not imply that the portfolio is efficient!)

Portfolio optimization

Here is a core problem we will consider: Given

- time-0 asset prices and capital $V(0)$
- time-T distribution of asset prices/return rates
- a target reward (mean time-T portfolio value)

determine a portfolio weight vector that minimizes squared risk.

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In return rates and with math: given statistics about $R_i(T)$ (possibly translated from those of $S_i(T)$) and a target expected return rate μ_P, compute $\boldsymbol{w}$ satisfying,

$$\min_{\boldsymbol{w}} \sigma_P^2 \quad \text{subject to} \quad \langle \boldsymbol{w}, \mathbf{1} \rangle = 1, \text{ and}$$
$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

$\mu = \mathbb{E}R$

$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mathbb{E}R_p$

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We can make this somewhat more transparent by realizing that σ_P^2 is a quadratic form of $\boldsymbol{w}$:

$$R_P = \langle \boldsymbol{R}, \boldsymbol{w} \rangle \implies \mu_P = \mathbb{E} R_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle,$$
$$\sigma_P^2 = \operatorname{Var} R_P = \mathbb{E}\left[R_P - \mu_P\right]^2 = \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w},$$

where

$$\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}, \qquad \boldsymbol{A} = \operatorname{Cov} \boldsymbol{R}.$$

Hence, given $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R})$, and $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$, then the optimization problem is now:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A}\boldsymbol{w} \;\; \text{subject to} \;\; \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1, \;\; \text{and}$$
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In the 2-security model, there are only two weights with two linear constraints, so actually there is typically no optimization to be done.

To be consistent with our model assumptions, we'll typically assume that $\boldsymbol{A}$ is positive-definite (no-arbitrage), and we'll allow $\boldsymbol{w}$ to have negative components (short selling).

Example

Consider the problem of constructing a (Markowitz) 2-security portfolio. The individual securities have expected return rates and covariance given by,

$$\mathbb{E}\boldsymbol{R}(T) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \text{Cov}\,\boldsymbol{R}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$\underline{\mu}$ $\underline{\underline{A}}$

For given μ_P, compute the optimal-risk portfolio.

$$\min_{\underline{w} \in \mathbb{R}^2} \underline{w}^T \underline{\underline{A}}\, \underline{w} \quad \text{subject to} \quad \underline{w}^T \underline{1} = 1, \quad \underline{w}^T \underline{\mu} = \mu_p$$

$$\min_{w_1, w_2} \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{s.t.} \quad w_1 + w_2 = 1, \quad 2w_1 + 6w_2 = \mu_p$$

Feasible set : $w_1 + w_2 = 1 \longrightarrow w_1 = 1 - w_2$

$2w_1 + 6w_2 = \mu_p \longrightarrow 2(1 - w_2) + 6w_2 = \mu_p$

$$w_2 = \frac{\mu_p - 2}{4}$$

$$w_1 = 1 - w_2 = \frac{6 - \mu_p}{4}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{4}\left[\begin{pmatrix} 6 \\ -2 \end{pmatrix} + \mu_p \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$= \underline{v}_0 + \mu_p \underline{v}_1 \quad , \quad \underline{v}_0 = \frac{1}{4}\begin{pmatrix} 6 \\ -2 \end{pmatrix} \quad , \quad \underline{v}_1 = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(optimal risk portfolio)

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For given μ_P, compute the optimal-risk portfolio.

What is the squared risk corresponding to this optimal portfolio?

$$\underline{w} = \underline{v}_0 + \mu_p \underline{v}_1 \qquad \underline{v}_0 = \frac{1}{4}\begin{pmatrix} 6 \\ -2 \end{pmatrix} \qquad \underline{v}_1 = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{Risk}^2 = \text{Var}\, R_p = \text{Var}\langle \underline{R}, \underline{w} \rangle = \underline{w}^T \underline{\underline{A}}\, \underline{w}$$

$$\text{Risk}^2 = (\underline{v}_0 + \mu_p \underline{v}_1)^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} (\underline{v}_0 + \mu_p \underline{v}_1) = \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0 + 2\mu_p \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1 + \mu_p^2 \underline{v}_1^T \underline{\underline{A}}\, v_1$$

$$16\, \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0 = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 12+2 \\ -6-4 \end{pmatrix} = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 14 \\ -10 \end{pmatrix}$$

$$= 84 + 20 = 104$$

$$\Rightarrow v_0^T A v_0 = \frac{104}{16} = \frac{26}{4} = 13/2$$

$4 \underline{\underline{A}}\, v_1$

$$16\, \underline{v}_1^T \underline{\underline{A}}\, \underline{v}_1 = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -2-1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$= 3 + 3 = 6$$

$$v_1^T A v_1 = 6/16 = 3/8$$

$$16\, \underline{v}_0^T \underline{\underline{A}}\, v_1 = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 & -2 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = -18 - 6$$

$$= -24$$

$$\underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1 = -24/16 = -3/2$$

$$Risk^2 = \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_0 + \left(2\, \underline{v}_0^T \underline{\underline{A}}\, \underline{v}_1\right) \mu_p + \left(\underline{v}_1^T \underline{\underline{A}}\, \underline{v}_1\right) \mu_p^2$$

$$= 13/2 - 3 \mu_p + 3/8\, \mu_p^2$$

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For given μ_P, compute the optimal-risk portfolio.

What is the squared risk corresponding to this optimal portfolio?

Compute the corresponding optimal risks for $\mu_P = 3$, and for $\mu_P = 5$.

$$\text{Risk}^2 = \frac{3}{8}\mu_p^2 - 3\mu_p + \frac{13}{2}$$

$$@\ \mu_p = 3: \ \text{Risk}^2 = \frac{3 \cdot 9}{8} - 9 + \frac{13}{2} = \frac{27}{8} - \frac{5}{2} = 7/8$$

@ $\mu_p = 5$: $Risk^2 = \frac{3 \cdot 25}{8} - 15 + \frac{13}{2} = \frac{75}{8} - \frac{17}{2} = \frac{75 - 68}{8} = 7/8$

($\mu_p = 3 \rightarrow$ inefficient portfolio)

A summary

For Markowitz 2-security portfolio optimization:

- We can formulate an optimization that minimizes risk at a given expected return rate.
- There is typically only one feasible portfolio, hence it's optimal.
- The resulting optimized risk can be *much* lower than the individual security risks.
- The optimal-risk portfolio is *not* necessarily efficient: there can be portfolios having the same risk but higher expected return!

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Some additional nomenclature:

- Markowitz portfolio theory is sometimes called "Modern" Portfolio Theory
- Often "risk" (or variance) is called *volatility*

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.