Math 5760/6890: Introduction to Mathematical Finance

2-Security Markowitz Efficient Frontier

See Petters and Dong 2016, Section 3.2

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A quick recap

For Markowitz 2-security portfolio optimization:

- Return rates $\boldsymbol{R}$ for the two securities are random variables
- Assume first- and second-order statistics of these are available: $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{R}$ and $\mathrm{Cov}(\boldsymbol{R})$
- The portfolio is defined by the weights $\boldsymbol{w}$
- The expected return rate of the portfolio is $\mu_P = \langle \boldsymbol{\mu}, \boldsymbol{w} \rangle$
- The squared risk of the portfolio is $\sigma_P^2 = \mathrm{Var}\langle \boldsymbol{R}, \boldsymbol{w} \rangle = \boldsymbol{w}^T \mathrm{Cov}(\boldsymbol{R})\boldsymbol{w}$

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We typically seek a risk-optimal portfolio for a given target mean μ_P:

$A = \mathrm{Cov}(R)$

$$\min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A}\boldsymbol{w} \;\text{ subject to }\; \langle \boldsymbol{w}, \boldsymbol{1} \rangle = 1, \;\text{ and}$$

$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P.$$

Lessons from last time

- We can formulate an optimization that minimizes risk at a given expected return rate.
- There is typically only one feasible portfolio, hence it's optimal.
- The resulting optimized risk can be *much* lower than the individual security risks.
- The optimal-risk portfolio is *not* necessarily efficient: there can be portfolios having the same risk but higher expected return!

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Today: efficient portfolios and the efficient "frontier".

Picking up from an example last time

Example

Consider the problem of constructing a (Markowitz) 2-security portfolio. The individual securities have expected return rates and covariance given by,

$$\mathbb{E}\boldsymbol{R}(T) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \qquad \mathrm{Cov}\,\boldsymbol{R}(T) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We've computed the risk-optimal portfolio as a function of μ_P:

$$\boldsymbol{w} = \begin{pmatrix} \frac{6-\mu_P}{4} \\ \frac{\mu-2}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{\mu_P}{4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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And we've used this to determine the optimal risk σ_P for a fixed μ_P:

$$\sigma_P^2 = \frac{3}{8}\mu_P^2 - 3\mu_P + \frac{13}{2}$$

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This relationship can be used to identify the set of all possible risk-optimal portfolios as a graph of valid (σ_P, μ_P) pairs.

$$\sigma_p^2 - \frac{3}{8}\mu_p^2 + 3\mu_p = \frac{13}{2}$$

$$\sigma_p^2 - \frac{3}{8}\left(\mu_p^2 - 8\mu_p\right) = \frac{13}{2}$$

$$\sigma_p^2 - \frac{3}{8}\left(\mu_p^2 - 8\mu_p + 16\right) = \frac{13}{2} - 6$$

$$\sigma_p^2 - \frac{3}{8}(\mu_p - 4)^2 = \frac{1}{2}$$

$$\sigma_p^2 - \frac{(\mu_p - 4)^2}{8/3} = \frac{1}{2}$$

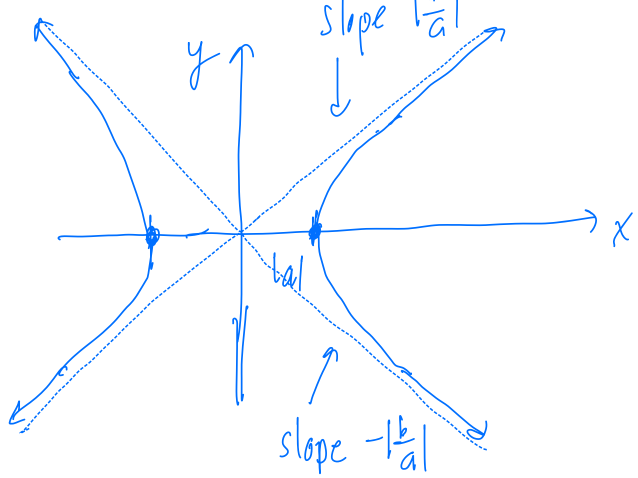
$$\frac{\sigma_p^2}{1/2} - \frac{(\mu_p - 4)^2}{4/3} = 1$$

$$\text{"}\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ \text{"}$$

$y = 0 \rightarrow x = \pm\sqrt{a^2} = \pm|a|$

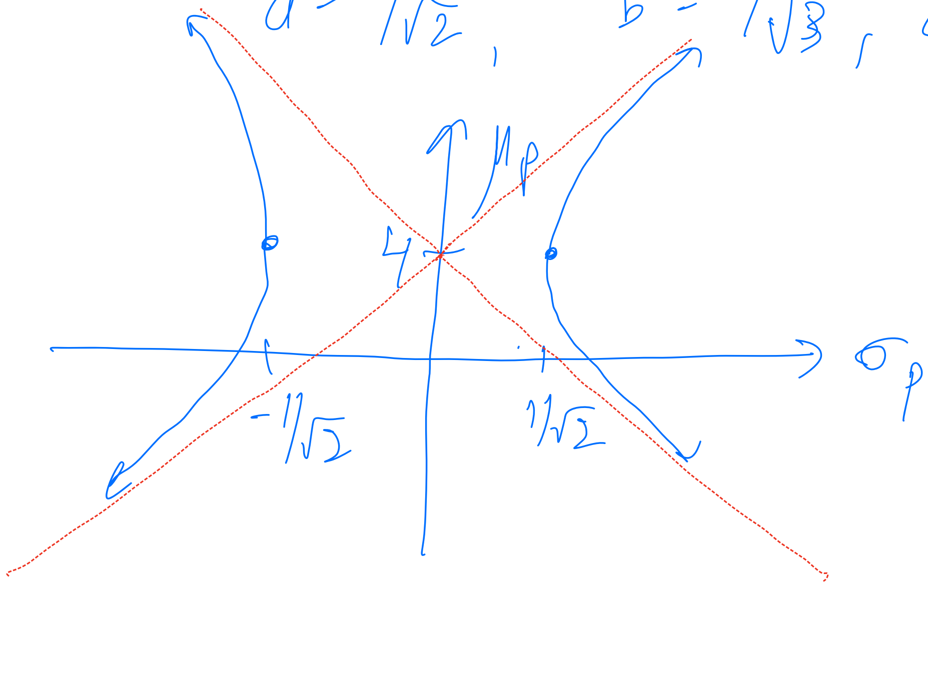
x, y very large : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightsquigarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$

$|y| = |\frac{b}{a} x|$

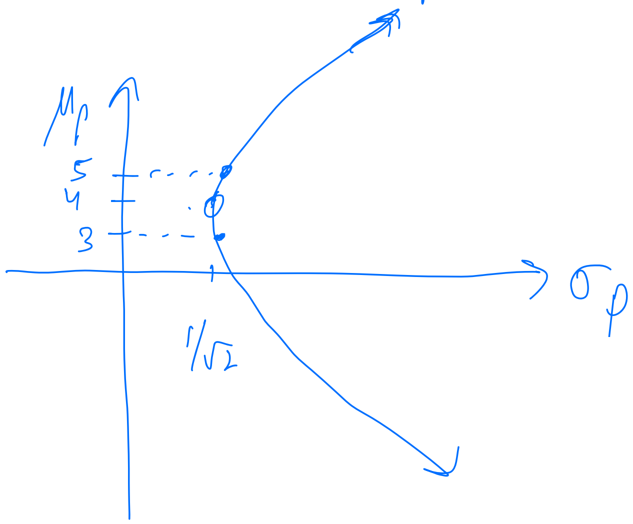


Back to $\frac{\sigma_p^2}{1/2} - \frac{(\mu_p - 4)^2}{4/3} = 1$

$a = 1/\sqrt{2}$, $b = 2/\sqrt{3}$, $\frac{b}{a} = \frac{2}{\sqrt{3}}\sqrt{2}$



No negative variances: $\sigma_p < 0$ irrelevant.



o: "global" variance-minimizing portfolio

$\mu_p = 4$

$\underbrace{\underline{w} \text{ @ } \mu_p = 4}_{\underline{w} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}$ $\quad \sigma_p = 1/\sqrt{2}$

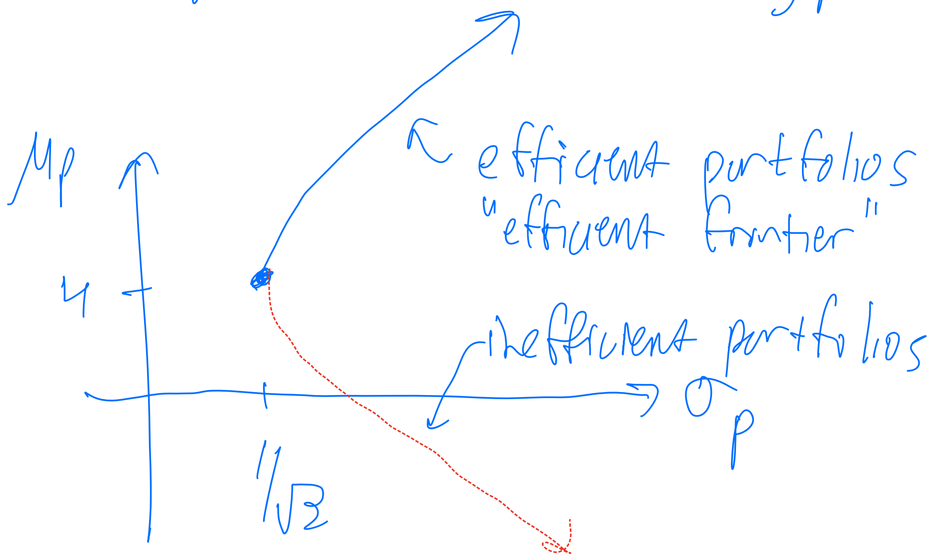
Which portfolios are efficient?

Efficient portfolio: μ_p increases $\Rightarrow \sigma_p$ increases

σ_p decreases $\Rightarrow \mu_p$ decrease

"In math": $\frac{d\sigma_p}{d\mu_p} > 0$ $\frac{d\mu_p}{d\sigma_p} > 0$

$\Longrightarrow$ Efficient portfolios are the "upper half" of the hyperbola



Efficient frontiers

The particular aspects we've explored in the previous example are generic:

- The set of risk-optimal portfolios is defined by the graph of a hyperbola in the (σ_P, μ_P) plane.
- The upper half of this graph are efficient portfolios – the *efficient frontier*.

Efficient frontiers

The particular aspects we've explored in the previous example are generic:

- The set of risk-optimal portfolios is defined by the graph of a hyperbola in the (σ_P, μ_P) plane.
- The upper half of this graph are efficient portfolios – the *efficient frontier*.
- There is a *global* risk-optimal portfolio, which corresponds to a particular expected return rate.
- In the (general) 2-security model, any feasible portfolio is risk-optimal.[1]

[1] This is not true if $\mu_1 = \mu_2$.

The general 2-security case

The previous aspects hold in the general 2-security case, assuming $\mu_1 = \mu_2$ and a positive-definite covariance.

Example

Consider the general 2-security setup:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \boldsymbol{A} = \mathrm{Cov}(\boldsymbol{R}),$$

where $\boldsymbol{A}$ is positive-definite and $\mu_1 \neq \mu_2$. Show that risk-optimal portfolios are given by the locus of points (σ_P, μ_P) satisfying,

$$\frac{\sigma_P^2}{a^2} - \frac{(\mu_P - \mu_G)^2}{b^2} = 1.$$

where a, b, μ_G are explicit constants: μ_G is the expected return rate of the global risk-optimal portfolio.

weight vectors are unique

$w_1 + w_2 = 1$

$\mu_1 w_1 + \mu_2 w_2 = \mu_P$

$$\Rightarrow \quad \underline{w} = \frac{1}{\mu_2-\mu_1}\begin{pmatrix} \mu_2-\mu_p \\ \mu_p-\mu_1 \end{pmatrix} = \underline{v_0} + \mu_p \underline{v_1}$$

$$\underline{v_0} = \frac{1}{\mu_2-\mu_1}\begin{pmatrix} \mu_2 \\ -\mu_1 \end{pmatrix}$$

$$\underline{v_1} = \frac{1}{\mu_2-\mu_1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\sigma_p^2 = \underline{w}^T \underline{\underline{A}}\, \underline{w}$$

$$= (\underline{v_0} + \underline{v_1}\mu_p)^T \underline{\underline{A}} (\underline{v_0} + \underline{v_1}\mu_p)$$

$$= \mu_p^2 (\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}) + 2\mu_p (\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}) + \underline{v_0}^T \underline{\underline{A}}\, \underline{v_0}$$

$$\sigma_p^2 = \underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}\left(\mu_p^2 + \frac{2\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}\mu_p\right) + \underline{v_0}^T \underline{\underline{A}}\, \underline{v_0}$$

$$= \underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}\left(\mu_p^2 + \frac{2\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}\mu_p + \left(\frac{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}\right)^2\right) + \underline{v_0}^T \underline{\underline{A}}\, \underline{v_0} - \frac{(\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0})^2}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}$$

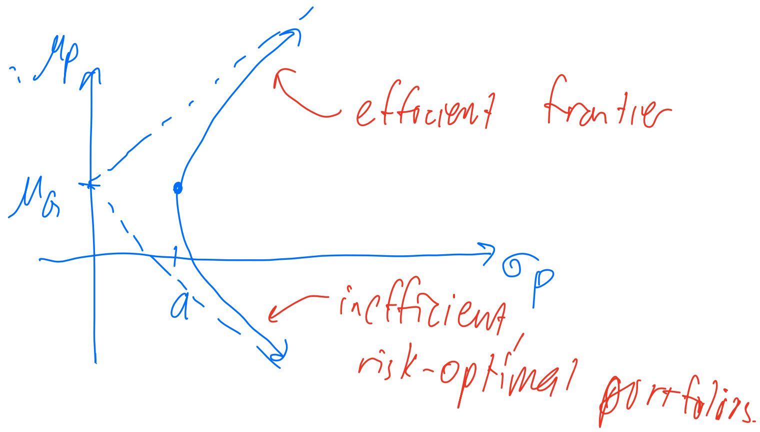
$$= \underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}\left(\mu_p + \frac{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}\right)^2 + \underline{v_0}^T \underline{\underline{A}}\, \underline{v_0} - \frac{(\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0})^2}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}$$

$$\Rightarrow \quad \frac{\sigma_p^2}{a^2} - \frac{(\mu_p - \mu_0)^2}{b^2} = 1, \qquad \mu_0 = \frac{-\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0}}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}$$

$$a^2 = \underline{v_0}^T \underline{\underline{A}}\, \underline{v_0} - \frac{(\underline{v_1}^T \underline{\underline{A}}\, \underline{v_0})^2}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}} \quad (>0)$$

$$b^2 = \frac{a^2}{\underline{v_1}^T \underline{\underline{A}}\, \underline{v_1}}$$

Plot risk-optimal portfolios:



What about short selling?

For general $\mathbb{E}\boldsymbol{R}$, $\mathrm{Cov}(\boldsymbol{R})$, μ_P, the risk-optimal portfolio could be highly leveraged (even if it's efficient).

In practice, we may want to disallow short selling to avoid such situations.

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In practice, we may want to disallow short selling to avoid such situations.

The corresponding optimization problem is an augmentation of what we've already seen:

$$\begin{aligned} \min_{\boldsymbol{w}} \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} \text{ subject to } \langle \boldsymbol{w}, \boldsymbol{1} \rangle &= 1, \text{ and} \\ \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle &= \mu_P, \text{ and} \\ w_i &\geqslant 0, \quad i = 1, \ldots, N. \end{aligned}$$

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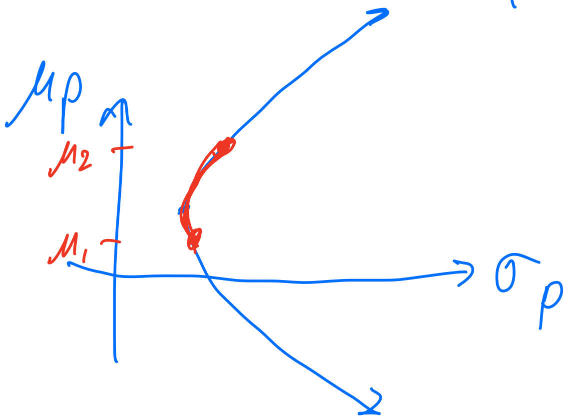
$$\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle = \mu_P, \ \text{and}$$
$$w_i \geqslant 0, \quad i = 1, \ldots, N.$$

(2-security case)

In general, this is a harder optimization problem to solve.

But in the 2-security case, the solution is fairly transparent.



2-security portfolio summary

In the 2-security case:

- Generally, no optimization is needed for 2-security portfolios: the weight constraints uniquely identify portfolios
- Many risk-optimal portfolios are *not* efficient
- The set of risk-optimal portfolios can be explicitly parameterized + plotted
- There is a global variance-minimizing portfolio – investors might not want this portfolio.

Moving to N-security portfolios

Many of the lessons we've learned from 2-security portfolios will hold in the N-security case:

- Risk-optimal portfolios can be computed and plotted
- The efficient frontier is visually identifiable
- There is a globally risk-optimal portfolio

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Many of the lessons we've learned from 2-security portfolios will hold in the N-security case:

- Risk-optimal portfolios can be computed and plotted
- The efficient frontier is visually identifiable
- There is a globally risk-optimal portfolio

There are some differences that make things more complicated:

- Optimization is generally required
- There are (many) feasible portfolios that are not risk-optimal
- Explicit pen+paper computations become harder

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.