Math 5760/6890: Introduction to Mathematical Finance

Security Pricing Models

See Petters and Dong 2016, Section 5.1

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The remainder of this course focuses on models for pricing securities (and what we can subsequently do with these models).

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The overall goal:

Given a security's time-0 price $S(0)$, construct a probabilistic model for $S(t)$, $t > 0$.

To accomplish this, we'll need to embed some information about how we expect $S(t)$ to behave in the future, such as its mean and variance.

The basic Bernoulli assumption

The underlying idea for this pricing model is as follows:

Consider a 1-period setup, so that we are building a model for $S(T)$, $T > 0$.

Our model for $S(T)$ will be that with some probability p, $S(T)$ increases in value, and with probability $1 - p$, it decreases in value.

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Hence $S(T)$ is a random variable, with probability mass function given by,

$$p_{S(T)}(s) = \begin{cases} p, & s = uS(0) \\ 1 - p, & s = dS(0) \end{cases}$$

$(u > 1)$
$(d < 1)$

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I.e.,

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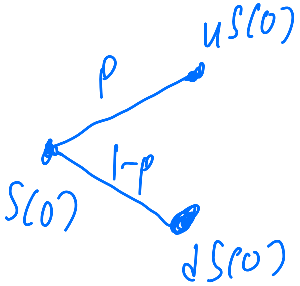

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An alternative (equivalent) point of view: we model the ratio $S(T)/S(0)$ as the random variable,

$$G := \frac{S(T)}{S(0)} = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1 - p \end{cases}$$

The quantity G is called the *gross return rate*.

Relative return: $\frac{S(t) - S(0)}{S(0)}$

$$S(T) = \begin{cases} uS(0), & \text{with probability } p \\ dS(0), & \text{with probability } 1-p \end{cases}$$

Thus, $S(T)$ is a (scaled version of a) **Bernoulli** random variable.

More precisely, $S(T)$ is determined by the outcome of a single biased(p) coin flip.

"Typical" Bernoulli RV's have a distribution:

$$X = \begin{cases} 1, & \text{w/prob } p \\ 0, & \text{w/prob } 1-p \end{cases}$$

$$Y = X\left(uS(0) - dS(0)\right) + dS(0)$$

has the same distribution as $S(T)$.

The variable $S(T)$

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Clearly for this model to be useful we should have:

- $u > 1$ (the asset could increase in value)
- $d < 1$ (the asset could decrease in value)
- $0 < p < 1$

} no-arbitrage

The triple (p, u, d) should depend on, for example, the predicted statistics of $S(T)$.

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This model in isolation has limited utility: it's a 1-period model.
(And is substantially less sophisticated than our portfolio analysis models.)

However, as a building block in a larger scheme, this is a very powerful model.

Example

Consider a one-period model for the price $S(T)$ of a security. The time-0 price $S(0)$ is known, and the predicted mean and variance at time T is (μ, σ^2). Compute the parameters (u, d) required so that a symmetric Bernoulli model for $S(T)$ (i.e., $p = 1/2$) yields the target mean and variance.

μ, σ^2 known, u, d unknown.

$$\mathbb{E}\, S(T) = \mu$$
$$\operatorname{Var} S(T) = \sigma^2$$

$$S(T) = \begin{cases} u S(0), & \text{with prob } 1/2 \\ d S(0), & \text{with prob } 1/2 \end{cases}$$

$$\mathbb{E}\, S(T) = \tfrac{1}{2} S(0) [u + d] = \mu \quad \longrightarrow \quad u + d = \frac{2\mu}{S(0)} \quad \longrightarrow \quad \mu = \frac{S(0)}{2}(u+d)$$

$$\operatorname{Var} S(T) = \mathbb{E}[S(T) - \mu]^2, \qquad Y = S(T) - \mu = \begin{cases} u S(0) - \mu & \text{w/prob } 1/2 \\ d S(0) - \mu, & \text{w/prob } 1/2 \end{cases}$$

$$Y = S(T) - \mu = \begin{cases} \frac{S(0)}{2}(u-d), & \text{w/prob } 1/2 \\ \frac{S(0)}{2}(d-u), & \text{w/prob } 1/2 \end{cases}$$

$$\operatorname{Var} S(T) = \mathbb{E}(S(T)-\mu)^2 = \mathbb{E}Y^2 = \frac{1}{2}\frac{S(0)^2}{4}(u-d)^2 + \frac{1}{2}\frac{S(0)^2}{\not{2}\,4}(d-u)^2$$

$$= \frac{S(0)^2}{4}(u-d)^2 = \sigma^2$$

$$u - d = \frac{2\sigma}{S(0)}$$

$$\left.\begin{aligned} u + d &= 2\mu/S(0) \\ u - d &= 2\sigma/S(0) \end{aligned}\right\} \quad 2u = \frac{2(\mu+\sigma)}{S(0)}$$

$$u = \frac{\mu+\sigma}{S(0)}$$

$$d = u - \frac{2\sigma}{S(0)} = \frac{\mu-\sigma}{S(0)}$$

A 2-period model

This one-step Bernoulli model serves as a building block for a 2-period model:

Let (p, u, d) be given. Introduce the random variable G having a shifted Bernoulli distribution:

$$p_G(g) = \begin{cases} p, & g = u \\ 1 - p, & g = d \end{cases}$$

Let G_1 and G_2 be two independent and identically distributed (“iid”) copies of G.

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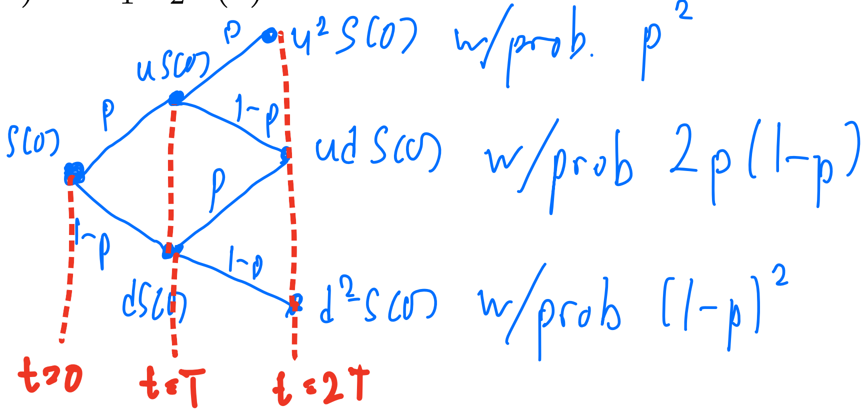
Our two-period model will define the trajectory of $S(t)$ as:

$$S(T) = S(0)G_1, \qquad S(2T) = S(T)G_2.$$

I.e., $S(T) = G_1 G_2 S(0)$. This can be visualized as a 4-leaf tree.



Some notation and terminology

Our 2-period model easily generalizes to the multi-period case, but before doing that we'll introduce some additional notation.

First, we'll now assume that $T > 0$ is a fixed terminal time (say 1 year), and our multi-period model divides this terminal time up into $n \in \mathbb{N}$ periods:

- A single period is of time length $h := T/n$.
- $t_j := jh = \frac{jT}{n}$ is the time after conclusion of the jth period.
- $t_0 = 0$, and $t_n = T$.
- We'll abbreviate $S(t_j) = S_j$ for $0 \leqslant j \leqslant n$.

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For a security $S(t)$, its *gross return* (rate) at time T is given by

$$G := \frac{S(T)}{S(0)}.$$

(Recall that the "standard" or capital-gain return rate is $R = G - 1$.)

The *log-return* of $S(t)$ at time T is,

$$L := \log G.$$

Our multi-period model can now be introduced: Let S be a security whose value at time 0, $S(0)$, is known. We seek to model the price $S(T)$ for $T > 0$ using an n-period model.

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We can write the time-n price as a product of inter-period gross returns:

$$S(T) = S_n = S_0 \frac{S_1}{S_0} \cdots \frac{S_n}{S_{n-1}} = S_0 \prod_{j=1}^{n} G_j, \qquad G_j := \frac{S_j}{S_{j-1}}.$$

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For our tree based model: we assume that $\{G_j\}_{j=1}^n$ are iid copies of a (scaled) Bernoulli random variable G, having distribution: (mass function)

$$p_G(g) = \begin{cases} p, & g = u \\ 1 - p, & g = d \end{cases}$$

where we assume knowledge of (p, u, d) satisfying our previously-mentioned assumptions.

Hence in this model: gross return rates between periods are independent but behave identically (have identical distributions).

Log-returns in the multi-period model

Since we have

$$S_n = S_0 \prod_{j=1}^{n} G_j,$$

then this immediately implies that the log-return over the entire time interval T is the sum of the inter-period log-returns:

$$L = \log \frac{S_n}{S_0} = \sum_{j=1}^{n} \log G_j =: \sum_{j=1}^{n} L_j.$$

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This multi-period model is multiplicative in the security price, but additive in the log-return.

And note that the L_j are also iid, with distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p, \\ \log d, & \text{with probability } 1-p. \end{cases}$$

Hence, the L_j are also (scaled versions of) Bernoulli random variables, and the cumulative log-return is a sum of iid Bernoulli random variables.

Outcomes

This model:

- Specifies the random distribution of $S(T)$ through n intermediate steps/periods.
- The relative (ratio) change between successive periods has a fixed distribution.
- All changes between periods are independent of each other.

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- Specifies the random distribution of $S(T)$ through n intermediate steps/periods.
- The relative (ratio) change between successive periods has a fixed distribution.
- All changes between periods are independent of each other.
- The tuple (p, u, d) describes the distribution of the gross return between periods.
- This results in a relatively complicated model: there are 2^n possible outcomes for n periods.
- Some (many) of the 2^n paths result in the same end price of $S(T)$. (n+1 outcomes)
- At some point we'll want to relate the (standard) return rate r of $S(T)$ to (p, u, d).
- Next time: $S(T)$ has a **Binomial** random variable distribution. Hence, this model is called the *Binomial Pricing Model*.

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.