Math 5760/6890: Introduction to Mathematical Finance

Binomial Pricing Models

See Petters and Dong 2016, Section 5.1

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The tree pricing model

The overall goal:

Given a security's time-0 price $S(0)$, construct a probabilistic model for $S(t)$, $t > 0$.

We do this with discrete time steps:

- Divide $[0, T]$ into $n \in \mathbb{N}$ equally sized intervals, $[t_j, t_{j+1}]$ for $j = 0, \ldots, N$.
- $t_j = jh$ with $h = T/n$
- Let $S_j = S(t_j)$
- Model $S_j \mapsto S_{j+1}$ as a multiplicative process:

uS_j

$$S_{j+1} = G_{j+1} S_j = \begin{cases} u\cancel{S_n}, & \text{with probability } p \\ d\cancel{S_n}, & \text{with probability } 1-p \end{cases}$$

dS_j

- We assume (p, u, d) satisfies $p \in (0, 1)$, and $d < 1 < u$.

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- We assume (p, u, d) satisfies $p \in (0, 1)$, and $d < 1 < u$.
- The behavior over the entire period corresponds to an accumulation of multiplicative gross returns G_j:

$$S_n = S_0 \prod_{j=1}^{n} G_j, \qquad L := \log \frac{S_n}{S_0} = \sum_{j=1}^{n} \log G_j = \sum_{j=1}^{n} L_j$$

- The random variables $\{G_j\}_{j=1}^{n}$ are iid (shifted) Bernoulli random variables (Same for $\{L_j\}_{j=1}^{n}$.)
- Today: we'll describe more details about this model.

A digression: Binomial random variables

Let $X \sim \mathrm{Bernoulli}(p)$, and consider n iid copies of X, $\{X_j\}_{j=1}^n$.

The random variable X satisfies,

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

X is the outcome of a (binary) coin toss, where the outcome is biased if $p \neq 1/2$.

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$\mathbb{E}X = p \cdot 1 + (1-p) \cdot 0 = p$

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Some simple first- and second-order statistics are immediately computable:

$$\mathbb{E}Y = \sum_{j=1}^{n} \mathbb{E}X_j = \sum_{j=1}^{n} p = np, \qquad \text{Var}Y \overset{(*)}{=} \sum_{j=1}^{n} \text{Var}X_j = np(1-p)$$

where $(*)$ uses the fact that the variance of the sum of independent random variables is the sum of their variances.

$$\operatorname{Var} X = \mathbb{E}(X-p)^2 = p(1-p)^2 + (1-p)(0-p)^2$$

$$= p(1-p)[\ 1\ ] = p(1-p$$

The Binomial distribution

The distribution (mass function) of Y is,

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \leqslant k \leqslant n$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0! = 1$$

The Binomial distribution

The distribution (mass function) of Y is,

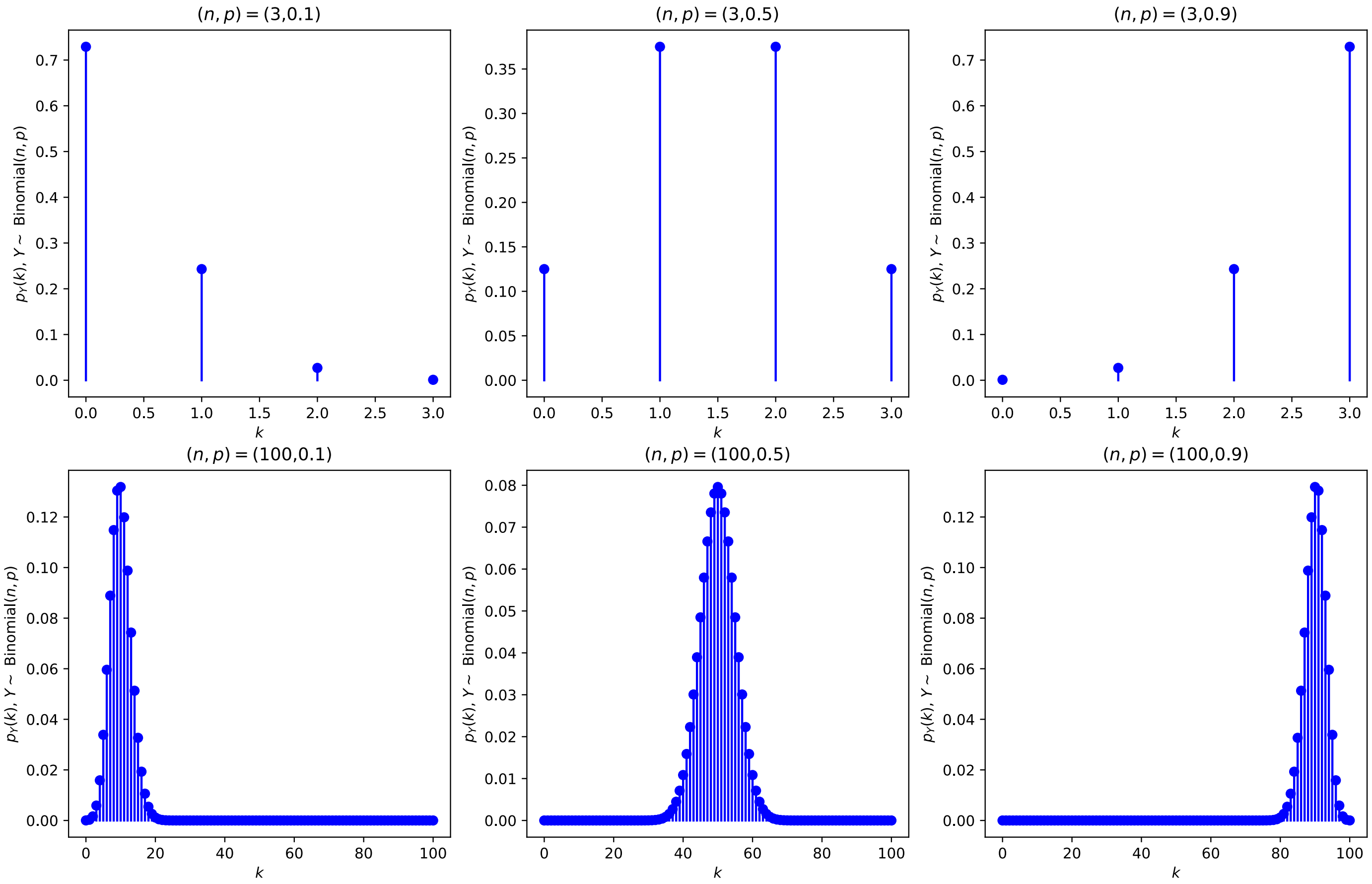
$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \leqslant k \leqslant n$$

This is a valid mass function due to the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \Longrightarrow \qquad \begin{aligned} \sum_{k=0}^{n} p_Y(k) &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \\ &= (p + (1-p))^n = 1. \end{aligned}$$

If $Y \sim$Binomial(n, p), the quantity $p_Y(k)$ is the probability that we observe exactly k "heads" outcomes from n independent coin flips biased to land on heads with probability p.

The Binomial distribution mass function



Shifting the Binomial distribution

The n Bernoulli trials need not have outcome $+1$ or 0.

Suppose that the Bernoulli-type random variable $\widetilde{X}$ has distribution,

$$\widetilde{X} = \begin{cases} b, & \text{with probability } p \\ a, & \text{with probability } 1-p \end{cases}$$

for any numbers a, b. Then note that if we write

$$\widetilde{X} = (b-a)X + a,$$

then $X \sim \mathrm{Bernoulli}(p)$.

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Then if we consider a sum of n iid copies of $\widetilde{X}$:

$$\widetilde{Y} := \sum_{j=1}^{n} \widetilde{X}_j, \qquad\qquad \widetilde{X}_j \stackrel{\text{iid}}{\sim} X,$$

then we observe that,

$$\widetilde{Y} = na + (b-a)Y,$$

where $Y \sim$Binomial(n, p) is a ("standard") Binomial random variable.

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Hence: iid sums of scaled/shifted Bernoulli random variables are just scaled/shifted Binomial random variables.

Recall that the log-return in our security model satisfies,

$$L = \sum_{j=1}^{n} L_j,$$

with L_j iid, having distribution,

$$L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Back to pricing

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$\log(u/d) = \log u - \log d$

Hence, we essentially know everything about L:

- L_j is a shifted Bernoulli(p) random variable. Precisely, $L_j = \log d + X_j \log \frac{u}{d}$, where $X_j \sim \text{Bernoulli}(p)$.
- L is a shifted Binomial(n,p) random variable: $L = n \log d + Y \log \frac{u}{d}$, where $Y \sim \text{Binomial}(n,p)$.

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Some caveats: S_n is not a Binomial random variable. Recall that $S_n = S_0 \exp L$.
The problem is that e^L is not Binomial (although its outcomes have the same probabilities as a Binomial random variable, they are not equispaced outcomes).

Although S_n is not Binomial, it's easy to use knowledge of the Binomial distribution to analyze outcomes.

Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

– Compute the mean and variance of the terminal-time log-return.

$n = 10$

$$L = \sum_{j=1}^{10} L_j, \quad L_j = \begin{cases} \log 1.1 & \text{w/prob } 0.6 \\ \log 0.9 & \text{w/prob } 0.4 \end{cases}$$

$$L_j = \log 0.9 + X_j \log\left(\frac{1.1}{0.9}\right), \quad X_j \sim \text{Bernoulli}(0.6)$$

$$\mathbb{E} L_j = \log 0.9 + \underbrace{(\mathbb{E} X_j)}_{p} \log\left(\frac{1.1}{0.9}\right) = \log 0.9 + 0.6 \log\left(\frac{1.1}{0.9}\right)$$

$$\mathbb{E} L = \sum_{j=1}^{10} \mathbb{E} L_j = 10 \cdot \left[\log 0.9 + 0.6 \cdot \log\left(\frac{1.1}{0.9}\right)\right] = 0.1504\ldots$$

$$\mathrm{Var}\, L = \mathrm{Var} \sum_{j=1}^{10} L_j = \sum_{j=1}^{10} \mathrm{Var}\, L_j$$

variance of sum of iid variables is the sum of variances

$$\mathrm{Var}\, L_j = \mathrm{Var}\left[0.9 + X_j \log\left(\frac{1.1}{0.9}\right)\right] = \log^2\left(\frac{1.1}{0.9}\right) \underbrace{\mathrm{Var}\, X_j}_{p(1-p)}$$

$$= \log^2\left(\frac{1.1}{0.9}\right)(0.6)(0.4)$$

$$\Longrightarrow \underline{\mathrm{Var}\, L} = 10 \cdot \mathrm{Var}\, L_j = \underline{0.4816}.$$

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Example

Consider a 10-period Binomial pricing model, with $(p, u, d) = (0.6, 1.1, 0.9)$.

- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?

$S_{10} = S_0 \cdot G$ $\quad$ G is a discrete RV with probabilities given by a Binomial$(10, 0.6)$ RV, with values $u^k d^{10-k}$, for $k=0, \ldots 10$.

$$\mathbb{E} S_{10} = S_0 \mathbb{E} G = S_0 \Big[u^0 d^{10} \underbrace{p^0 (1-p)^{10} \binom{10}{0}}_{\text{Binomial}(10,\,0.6)\ \text{prob.}} + u^1 d^9 p^1 (1-p)^9 \binom{10}{1} + \ldots \Big]$$

$$= S_0 \sum_{k=0}^{10} u^k d^{10-k} \, p^k (1-p)^{10-k} \binom{10}{k} \qquad (p, u, d) = (0.6, 1.1, 0.9)$$

$$= 1.219 (S_0).$$

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- Compute the mean and variance of the terminal-time log-return.
- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?

$$S_{10} = S_0 \cdot u^X d^{10-X} \quad , X: \text{ \# "heads" outcome.}$$

$$= S_0 \left(\frac{u}{d}\right)^X d^{10}$$

When is $\left(\frac{u}{d}\right)^X d^{10} \geq 1$?

what's the smallest x s.t. $\left(\frac{1.1}{0.9}\right)^x (0.9)^{10} \geq 1$?

$x = 6$

$\Rightarrow$ $x = 6, 7, 8, 9, 10$ correspond to $S_{10} \geq S_0$.

$P(S_{10} \geq S_0) = P(x \geq 6) = \sum_{k=6}^{10} Pr(x=k) = \sum_{k=6}^{10} \binom{10}{k} (0.6)^k (0.4)^{10-k}$

$= 0.63$

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- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?
- What is the probability that $S_5 \geqslant S_0$?

$$S_5 = S_0 \left(\frac{1.1}{0.9}\right)^x (0.9)^5, \quad x = 0, 1, 2, .., 5$$

$$\left(\frac{1.1}{0.9}\right)^x (0.9)^5 \geq 1 \quad \text{when} \quad x \geq 3$$

$$\Rightarrow \Pr(S_5 \geq S_0) = \sum_{k=3}^{5} \binom{5}{k} 0.6^k (0.4)^{5-k} = 0.683$$

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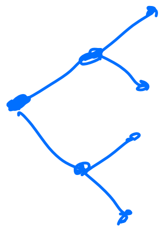
- What is the expected value of S_{10}?
- What is the probability that $S_{10} \geqslant S_0$?
- What is the probability that $S_5 \geqslant S_0$?
- Suppose $S_5 = S_0(1.1)^5$. What is the distribution of S_{10} conditioned on this outcome?

The distribution of S_{10} conditioned on $S_5 = (1.1)^5 S_0$ is the same as the distribution of a 5-period model with initial value $(1.1)^5 S_0$.

$$S_{10} \text{ conditioned on } S_5 = (1.1)^5 \cdot S_0 = \begin{cases} (1.1)^5 S_0 \, (1.1)^5, & \text{w/ prob } p^5 (1-p)^0 \binom{5}{0} \\ \vdots & \vdots \\ (1.1)^5 S_0 \, (0.9)^5 & \text{w/prob } p^0 (1-p)^5 \binom{5}{5} \end{cases}$$

(this model is "memoryless")

Some final observations



The process we have defined and investigated has some useful properties:

There are 2^n possible pricing trajectories that the model can take. (2 options at each time.)

But there are only $j+1$ possible outcomes at time t_j: the security price S_j is given by,

$$S_j = S_0 \prod_{q=1}^{j} G_j = S_0 u^{Y_j} d^{j-Y_j},$$

where Y_j is the cumulative number of "heads" realizations in the first j steps:

$$Y_j = \sum_{q=1}^{j} X_q.$$

Since Y_j has exactly $j+1$ possible outcomes, then S_j, whose randomness depends explicitly and only on Y_j, has exactly $j+1$ possible outcomes.

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Related to the above, the tree is *recombining*, meaning that there are trajectories that reach the same state using different pathways. In particular,

- outcome $(G_1, G_2) = (u, d)$
- outcome $(G_1, G_2) = (d, u)$

yield the same security price $S_2 = udS_0$.

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As we saw in the previous example, this is a *Markovian* process: the possibilities at time t_{j+1} can be deduced entirely from the state of things at time t_j.

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.