Math 5760/6890: Introduction to Mathematical Finance

Binomial Options Pricing

See Petters and Dong 2016, Section 5.1

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

Fall 2024

www.sci.utah.edu

The tree pricing model

We've seen the basic anatomy of the binomial pricing model: Given (p, u, d), then,

$$S_j = G_j S_{j+1}, \qquad\qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

where $S_j = S(t_j)$ and G_j is the gross return rate.

The tree pricing model

We've seen the basic anatomy of the binomial pricing model: Given (p, u, d), then,

$$\cancel{S_j = G_j S_{j+1},}$$

$$S_{j+1} = G_j S_j \qquad\qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

where $S_j = S(t_j)$ and G_j is the gross return rate. In this model, it turns out that *log returns* are particularly convenient to work with:

$$\frac{S_n}{S_0} = e^L, \qquad\qquad L = \sum_{j=1}^{n} L_j, \qquad\qquad L_j = \begin{cases} \log u, & \text{with probability } p \\ \log d, & \text{with probability } 1-p \end{cases}$$

Our (p, u, d) assumptions

Recall that we've always assumed,

- $p \in (0, 1)$
- $d < 1 < u$

The $p \in (0, 1)$ assumption is reasonable: if $p = 0, 1$, then the model is not random, implying that there is no uncertainty about the future.

Our (p, u, d) assumptions

Recall that we've always assumed,

- $p \in (0, 1)$
- $d < 1 < u$

The $p \in (0, 1)$ assumption is reasonable: if $p = 0, 1$, then the model is not random, implying that there is no uncertainty about the future.

The assumption that $u > d$ is just for convenience: if $u < d$, then consider another triple $(q, \tilde{u}, \tilde{d})$ with,

- $q = 1 - p$
- $\tilde{u} = d$
- $\tilde{d} = u < \tilde{u}$

Then the (p, u, d) is equivalent in distribution to the $(q, \tilde{u}, \tilde{d})$ model, but the latter satisfies $\tilde{u} > \tilde{d}$.

Hence, we assume $u > d$ without loss of generality.

Our (p, u, d) assumptions

Recall that we've always assumed,

- $p \in (0, 1)$
- $d < 1 < u$

The $p \in (0, 1)$ assumption is reasonable: if $p = 0, 1$, then the model is not random, implying that there is no uncertainty about the future.

The assumption that $u > d$ is just for convenience: if $u < d$, then consider another triple $(q, \tilde{u}, \tilde{d})$ with,

- $q = 1 - p$
- $\tilde{u} = d$
- $\tilde{d} = u < \tilde{u}$

Then the (p, u, d) is equivalent in distribution to the $(q, \tilde{u}, \tilde{d})$ model, but the latter satisfies $\tilde{u} > \tilde{d}$.

Hence, we assume $u > d$ without loss of generality.

Finally, the fact that 1 must be sandwiched between d and u is a requirement to ensure a no-arbitrage setup:

- If $d \geqslant 1$, then $\Pr(S_1 \geqslant S_0) = 1$, and $\Pr(S_1 > S_0) = p > 0$, ensuring an arbitrage by holding a long position in S.
- If $u \leqslant 1$, then $\Pr(S_1 \leqslant S_0) = 1$, and $\Pr(S_1 < S_0) = p > 0$, ensuring an arbitrage by holding a short position in S.

There is one more concept that will be useful for us to employ in modeling investors:

Suppose I seek to sell you an asset today at a price S_0.

If you have a probabilistic model for the future trajectory of $S(t)$, and if your model predicts $\mathbb{E}S(t) < S_0$, then you have limited incentive purchase this asset.

On the other hand, if your model (correctly!) predicts $\mathbb{E}S(t) > S_0$, then there is opportunity for arbitrage, assuming you have unlimited capital to invest.

In such a case, we assume that another saavy investor would have already recognized this and removed the arbitrage opportunity through exploitation; hence your model is unlikely to be accurate.

Risk neutrality

There is one more concept that will be useful for us to employ in modeling investors:

Suppose I seek to sell you an asset today at a price S_0.

If you have a probabilistic model for the future trajectory of $S(t)$, and if your model predicts $\mathbb{E}S(t) < S_0$, then you have limited incentive purchase this asset.

On the other hand, if your model (correctly!) predicts $\mathbb{E}S(t) > S_0$, then there is opportunity for arbitrage, assuming you have unlimited capital to invest.

In such a case, we assume that another saavy investor would have already recognized this and removed the arbitrage opportunity through exploitation; hence your model is unlikely to be accurate.

Based on these scenarios, then a reasonable assumption on a valid model is that $\mathbb{E}S(t) = S_0$.

A probabilistic model satisfying this assumption is said to be **risk neutral**.
(In more realistic scenarios, we'll want the *present value* of $\mathbb{E}S(t)$ to satisfy this.)

More formally, a risk-neutral probability measure satisfies the above assumption.

Note that for our tree model, we have

$$\mathbb{E}S_1 = S_0\mathbb{E}G_1 = S_0\left(pu + (1-p)d\right),$$

The risk-neutrality requirement is that $\mathbb{E}S_1 = S_0$.

$$\Rightarrow pu + (1-p)d = 1$$

$$p(u-d) = 1-d$$

$$p = \frac{1-d}{u-d}$$

Note that for our tree model, we have

$$\mathbb{E}S_1 = S_0 \mathbb{E}G_1 = S_0 \left(pu + (1-p)d\right),$$

The risk-neutrality requirement is that $\mathbb{E}S_1 = S_0$.

Hence, in order for this single-period jump to be risk-neutral, then we require,

$$p = \frac{1-d}{u-d}.$$

This prescribes p in terms of the single-period upward/downward factors.
(And note that assuming $d < 1 < u$ implies that $p \in (0,1)$.)

Risk neutrality in practice

Note that for our tree model, we have

$$\mathbb{E}S_1 = S_0 \mathbb{E}G_1 = S_0 \left(pu + (1-p)d \right),$$

The risk-neutrality requirement is that $\mathbb{E}S_1 = S_0$.

Hence, in order for this single-period jump to be risk-neutral, then we require,

$$p = \frac{1-d}{u-d}.$$

This prescribes p in terms of the single-period upward/downward factors.
(And note that assuming $d < 1 < u$ implies that $p \in (0,1)$.)

This one-period result extends to multiple periods through induction:

$$\mathbb{E}[S_{n+1} \,|\, S_n] = S_n \mathbb{E}G_n = S_n (pu + (1-p)d).$$

And again if p is as given by the formula above, then $\mathbb{E}S_{n+1} = \mathbb{E}S_n = S_0$.

Such a model has limitations – e.g., in practice one would not be interested in the security if the average return was 0.

In practice, we assume an average return rate m (in units matching those of t).

The idea: we should discount future values based on this return rate.

Such a model has limitations – e.g., in practice one would not be interested in the security if the average return was 0.

In practice, we assume an average return rate m (in units matching those of t).

The idea: we should discount future values based on this return rate.

Hence, assuming compounding corresponding to the number of periods in the model, then the present value of S_1 is,

$$\mathrm{PV}(S_1) = S_1 \left(1 + \frac{Tm}{n}\right) \overset{\Delta t = T/n}{=} S_1 \left(1 + m\Delta t\right),$$

so that the risk-neutral value of p in this case is,

$$p = \frac{(1 + m\Delta t) - d}{u - d}$$

Another practicality worth building in: frequently the asset S is a stock.

Many stocks pay regular dividends, corresponding to a rate q.

A stock that pays dividends at rate q should be discounted accordingly:

$$\mathrm{PV}(S_1) = S_1\left(1 + \frac{T\overset{m}{\cancel{r}}}{n} - \frac{Tq}{n}\right) \stackrel{\Delta t = T/n}{=} S_1\left(1 + (m-q)\Delta t\right),$$

Some real-world considerations, II

Another practicality worth building in: frequently the asset S is a stock.

Many stocks pay regular dividends, corresponding to a rate q.

A stock that pays dividends at rate q should be discounted accordingly:

$$\mathrm{PV}(S_1) = S_1 \left(1 + \frac{Tr}{n} - \frac{Tq}{n}\right) \stackrel{\Delta t = T/n}{=} S_1 \left(1 + (m - q)\Delta t\right),$$

The risk-neutral value of p in this case is,

$$p = \frac{(1 + (m - q)\Delta t) - d}{u - d}$$

Of course one expects that $m > q$ in order for the stock to be attractive to investors.

Note also that we must have

$$\Delta t < \frac{u}{m - q},$$

in order for this to be a valid model ($p < 1$).

Although we have not really discussed options too much, one of the main applications of this model is in the pricing of options.

We'll only identify some infrastructure right now without explicitly identifying a pricing model.
We'll derive an actual pricing model in a couple of weeks.

For simplicity, let's consider a European call option:

- At $t = 0$, we are buying the right (not the requirement) to purchase a (say single) share of S. ("call")
- We can exercise this option *only* at time $t = T$ to purchase the stock at strike price K. ("European")

The question: what price (premium) should we be willing to pay for this option?

Options pricing

Although we have not really discussed options too much, one of the main applications of this model is in the pricing of options.

We'll only identify some infrastructure right now without explicitly identifying a pricing model.
We'll derive an actual pricing model in a couple of weeks.

For simplicity, let's consider a European call option:

- At $t = 0$, we are buying the right (not the requirement) to purchase a (say single) share of S. ("call")
- We can exercise this option *only* at time $t = T$ to purchase the stock at strike price K. ("European")

The question: what price (premium) should we be willing to pay for this option?

Here is the basic logic of the options pricing model:

- We'll generate a probabilistic model for all time-T outcomes of the options price.
- We'll propagate these prices *backward* in time through a pricing model.
- The resulting time-0 price (a deterministic number) will be our modeled security price.

Although we have not really discussed options too much, one of the main applications of this model is in the pricing of options.

We'll only identify some infrastructure right now without explicitly identifying a pricing model.
We'll derive an actual pricing model in a couple of weeks.

For simplicity, let's consider a European call option:

- At $t = 0$, we are buying the right (not the requirement) to purchase a (say single) share of S. ("call")
- We can exercise this option *only* at time $t = T$ to purchase the stock at strike price K. ("European")

The question: what price (premium) should we be willing to pay for this option?

Here is the basic logic of the options pricing model:

- We'll generate a probabilistic model for all time-T outcomes of the options price.
- We'll propagate these prices *backward* in time through a pricing model.
- The resulting time-0 price (a deterministic number) will be our modeled security price.

We'll assume that the following information is available:

- stock volatility (typically through historical data)
- a risk-free rate r, and a dividend rate q

Options pricing, step 1: terminal time prices

Recall that the period-n outcomes of the binomial model are:

$$S_0 u^n d^0, \;\; S_0 u^{n-1} d^1 \;\; , \ldots, \;\; S_0 u^1 d^{n-1}, \;\; S_0 u^0 d^n.$$

We'll *assume* that (u, d) are given and prescribed.
(We'll soon see that a common model is to assign (u, d) based on the historic volatility of the stock.)

Recall that the period-n outcomes of the binomial model are:

$$S_0 u^n d^0, \;\; S_0 u^{n-1} d^1 \;\; , \ldots, \;\; S_0 u^1 d^{n-1}, \;\; S_0 u^0 d^n.$$

We'll *assume* that (u, d) are given and prescribed.
(We'll soon see that a common model is to assign (u, d) based on the historic volatility of the stock.)

The prices $\{\widetilde{S}_{n,j}\}_{j=0}^{n}$, with

$$\widetilde{S}_{n,j} := S_0 u^j d^{n-j},$$

are our modeled terminal prices of the stock.

Recall that the period-n outcomes of the binomial model are:

$$S_0 u^n d^0, \;\; S_0 u^{n-1} d^1 \;\;, \ldots, \;\; S_0 u^1 d^{n-1}, \;\; S_0 u^0 d^n.$$

We'll *assume* that (u, d) are given and prescribed.
(We'll soon see that a common model is to assign (u, d) based on the historic volatility of the stock.)

The prices $\{\widetilde{S}_{n,j}\}_{j=0}^n$, with

$$\widetilde{S}_{n,j} := S_0 u^j d^{n-j},$$

are our modeled terminal prices of the stock.

Note that the time-T value of the call option is the value of the call relative to the strike price:

$$\widehat{S}_{n,j} := \max\{0, \cancel{K - \widetilde{S}_{n,j}} \; \widetilde{S}_{n,j} - K\}$$

I.e., the stock has value $\cancel{K - \widetilde{S}_{n,j}}$ $\widetilde{S}_{n,j} - K$ should this difference be positive, but is 0 otherwise as we simply choose not to exercise the option.

Options pricing, step 2: back-propagation under a risk-neutral measure

We know values at period n. How might be propoagate these prices back to time $n-1$?

There are n values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that we must determine.

Using the binomial tree, we use risk neutrality:

- We define a risk-neutral measure by identifying p appropriately:

$$p = \frac{(1 + (m-q)\Delta t) - d}{u - d}$$

- The value $\widehat{S}_{n-1,j}$ should be the expected value of the time-n security under the risk-neutral measure:

$$\underbrace{\widetilde{S}_{n-1,j}}_{\text{"today's price"}} = \underbrace{p\widehat{S}_{n,j} + (1-p)\widehat{S}_{n,j+1}}_{\mathbb{E}\text{ of "tomorrow's" price}}, \qquad j = 0, \ldots, n$$

Options pricing, step 2: back-propagation under a risk-neutral measure

We know values at period n. How might be propoagate these prices back to time $n-1$?

There are n values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that we must determine.

Using the binomial tree, we use risk neutrality:

- We define a risk-neutral measure by identifying p appropriately:

$$p = \frac{(1 + (m-q)\Delta t) - d}{u - d}$$

- The value $\widehat{S}_{n-1,j}$ should be the expected value of the time-n security under the risk-neutral measure:

$$\widetilde{S}_{n-1,j} = p\widehat{S}_{n,j} + (1-p)\widehat{S}_{n,j+1}, \qquad j = 0, \ldots, n$$

- We should discount by the risk-free rate:

$$\widehat{S}_{n-1,j} = e^{-r\Delta t}\widetilde{S}_{n-1,j}.$$ [assuming no dividends]

(assuming for simplicity continuous compounding)

Options pricing, step 2: back-propagation under a risk-neutral measure

We know values at period n. How might be propoagate these prices back to time $n-1$?

There are n values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that we must determine.

Using the binomial tree, we use risk neutrality:

- We define a risk-neutral measure by identifying p appropriately:

$$p = \frac{(1+(m-q)\Delta t) - d}{u-d}$$

- The value $\widehat{S}_{n-1,j}$ should be the expected value of the time-n security under the risk-neutral measure:

$$\widetilde{S}_{n-1,j} = p\widehat{S}_{n,j} + (1-p)\widehat{S}_{n,j+1}, \qquad\qquad j = 0, \ldots, n$$

- We should discount by the risk-free rate:

$$\widehat{S}_{n-1,j} = e^{-r\Delta t}\widetilde{S}_{n-1,j}.$$

  (assuming for simplicity continuous compounding)

This results in values $\{\widehat{S}_{n-1,j}\}_{j=0}^{n-1}$ that are modeled prices of the option value at time $n-1$.

Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

An extra detail: different types of options have different rules of exercise.

In American options, one can exercise the option at any time up until time T.

Under an American option, the risk-neutral expectation should be modified not only by a present value discount, but also by the possibility of exercise:

– At period k, compute the standard binomial tree value of the stock at the current period (call this say $S_{k,j}$).

Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

An extra detail: different types of options have different rules of exercise.

In American options, one can exercise the option at any time up until time T.

Under an American option, the risk-neutral expectation should be modified not only by a present value discount, but also by the possibility of exercise:

- At period k, compute the standard binomial tree value of the stock at the current period (call this say $S_{k,j}$).
- The "exercise value" is $\max\{0, K - S_{k,j}\}$.

Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

An extra detail: different types of options have different rules of exercise.

In American options, one can exercise the option at any time up until time T.

Under an American option, the risk-neutral expectation should be modified not only by a present value discount, but also by the possibility of exercise:

- At period k, compute the standard binomial tree value of the stock at the current period (call this say $S_{k,j}$).
- The "exercise value" is $\max\{0, K - S_{k,j}\}$.
- Replace $\widehat{S}_{k,j}$ computed as before (the "Binomial value") with the maximum of the exercise and binomial value:

$$\widehat{S}_{k,j} \leftarrow \max\{\widehat{S}_{k,j}, \max\{0, K - S_{k,j}\}\}$$

Options pricing, step 3: iterate

One sequentially moves from time index $n \mapsto n-1 \mapsto n-2 \cdots \mapsto 1 \mapsto 0$.

At time index 0, there is a single value, and it is the (modeled) premium that one should be willing to pay for the option.

An extra detail: different types of options have different rules of exercise.

In American options, one can exercise the option at any time up until time T.

Under an American option, the risk-neutral expectation should be modified not only by a present value discount, but also by the possibility of exercise: (for put option)

- At period k, compute the standard binomial tree value of the stock at the current period (call this say $S_{k,j}$).
- The "exercise value" is $\max\{0, K - S_{k,j}\}$.
- Replace $\widehat{S}_{k,j}$ computed as before (the "Binomial value") with the maximum of the exercise and binomial value:

$$\widehat{S}_{k,j} \leftarrow \max\{\widehat{S}_{k,j}, \max\{0, K - S_{k,j}\}\}$$

The idea is that if exercise is possible at a certain time, we should model it.

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.