Math 5760/6890: Introduction to Mathematical Finance

The Cox-Ross-Rubinstein model

See Petters and Dong 2016, Section 5.2

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The binomial tree pricing model

We have modeled a security's price $S_j = S(t_j)$ via,

$$S_{j+1} = G_{j+1} S_j, \qquad G_j = \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

From this model, we've concluded:

- $L := \log(S_n/S_0)$ is a scaled/shifted $\mathrm{Binomial}(n,p)$ random variable.
- $S_n = S_0 e^L$ is the exponential of a scaled/shifted Binomial random variable
- The triple (p, u, d) determines the distribution entirely.

Properties of the model

It's worth pointing out some terminology that relates to our model

- The return is determined by the sequence of iid (scaled/shifted) Bernoulli random variables $\{L_j\}_{j=1}^n$. This sequence is called a scaled/shifted **Bernoulli process**.

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Hence, we have constructed a geometric random walk model for an asset's price.

How can we choose (p, u, d) to accurately emulate real stock price behavior?

We will follow the *Cox-Ross-Rubinstein* (CRR) model, which in part appeals to limiting continuous-time arguments to set the discrete-time values (p, u, d).

Degrees of freedom and constraints

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These are all reasonable assumptions, and one of the major appeals of the CRR Model is that these rather simple assumptions yield probabilistic models that behave like real-world stock prices.

A more abstract reason for these constraints is that they can be used to construct a well-posed underlying continuous-time mathematical model.

The recombination condition

The first CRR constraint is that,

$$ud = 1 \qquad \Longrightarrow \qquad d = \frac{1}{u}$$

One visually appealing result of this assumption is that the tree is has a symmetry around $S_n = S_0$. (It is easier to see this for log-returns.)

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- It's a sensible condition if we adopt the hypothesis that sequential upward and downward market movements are geometrically symmetric.
- I.e., it makes the log-return a symmetric (not necessarily centered) random walk:

$$\log L_j = \begin{cases} \log u, & \text{with probability } p, \\ -\log u, & \text{with probability } 1-p. \end{cases}$$

The mean-matching condition

We seek to impose that the average of the geometric random walk process emulate an expected return rate.

We *could* do this by using historical data to compute an inter-period average return. However, there are some problems with this:

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- Stock data typically computes average (say annual) return rate, which in other finance contexts is treated essentially as a continuous-time rate.
- It's somewhat awkward to understand what should happen when a more complicated model is desired:
  - Suppose we want the random model to have 4 coin flips per day.
  - We have to determine an appropriate quarter-day return rate.
  - This is not too difficult, but it's easier if we assume a continuous return rate and use that to determine discrete-time rates.

Calibrating the mean

Because our random walk is geometric, it's easier to calibrate the mean of the log-returns.

With T fixed, suppose we choose a number of equal periods n that are used to divide $[0, T]$.
From our previous notation: $h = h_n := T/n$.

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The mean of the log-return is given by:

ud=1

$$\mathbb{E}L_j = \mathbb{E}\log G_j = p\log u + (1-p)\log d = (2p-1)\log u$$

This mean is the expected (log-)return over a time period of length h_n.

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The assumption in a CRR model is that, as $n \to \infty$, μ_n converges to a constant value:

$$\lim_{n\to\infty} \mu_n = \mu_{\mathrm{RW}},$$

"RW": "real world"

i.e., $\mu_n \approx \mu_{\mathrm{RW}}$ for large n.

Estimating $\mu = \mu_{RW}$

Note that μ is in units of one over time. As with most rates, the unit of time used in practice is typically a year.

The constant μ is an instantaneous log-return: it is frequently called the real-world/instantaneous/continuous-time **drift**.

To approximate μ, we use data in order to realize the approximation,

$$\mu \approx \mathbb{E}\frac{L}{h} = \mathbb{E}\frac{1}{t_1 - t_0} \log \frac{S(t_1)}{S(t_0)}.$$

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Example

Suppose we are given (deterministic) daily security prices $S_0, S_1, S_2, \ldots, S_n$.
(E.g., this could come from historical data, and there are ≈252 trading days per year.)
Compute (an approximation to) the continuous-time drift.

A single sample of the (time-normalized) log-return is $\frac{1}{t_{j+1} - t_j} \log \frac{S_{j+1}}{S_j}$

$1 = 252$ periods $\Longrightarrow$ $h_n = \frac{1}{252} = t_{j+1} - t_j$ (assume 1 year w/ 252 samples)

$\parallel$

n

$$\Longrightarrow \left\{ \frac{1}{t_{j+1}-t_j} \log\left(\frac{S_{j+1}}{S_j}\right) \right\}_{j=0}^{251} = \left\{ 252 \log\left(\frac{S_{j+1}}{S_j}\right) \right\}_{j=0}^{252}$$

$\underbrace{t_{j+1}-t_j}$: $\frac{1}{\left(\frac{1}{252}\right)}$

empirical mean of these is an approximation (from data) to $\mu = \mu_{RW}$.

The variance-matching condition

The final CRR constraint is similar to the mean-matching condition: we assume a well-defined instantaneous rate of change for the variance.

The discrete-time variance, normalized by the time period, is

$$\sigma_n^2 := \frac{1}{h} \mathrm{Var} \log \frac{S_{j+1}}{S_j},$$

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The CRR model makes the assumption that $n \to \infty$, then σ_n^2 converges to a constant,

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Unlike μ, the limiting process for σ (not σ^2) scales like $1/\sqrt{n}$.

Just like the continuous-time drift μ, estimate the volatility σ is typically accomplished through access to data with the finite-time approximation,

$$\sigma^2 \approx \frac{1}{h}\mathrm{Var} L = \frac{1}{t_1 - t_0}\mathrm{Var} \log \frac{S(t_1)}{S(t_0)}.$$

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Compute (an approximation to) the continuous-time volatility.

In summary

The CRR model places the following additional constraints on our standard Binomial tree model:

- Geometric symmetry of tree prices: $u = 1/d$
- The continuous-time limit of the expected log-return matches the real-world drift:

$$\mu = \lim_{n \to \infty} \frac{1}{h_n} \mathbb{E} L_j$$

- The continuous-time limit of the variance of the log-return matches the real-world (squared) volatility:

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In the last two bullets above, both $\mathbb{E} L_j$ and $\mathrm{Var} L_j$ depend on (p, u, d).

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In the last two bullets above, both $\mathbb{E} L_j$ and $\mathrm{Var} L_j$ depend on (p, u, d).

Hence, for finite n, (p, u, d) should depend on the time discretization parameter n. I.e., for finite n,

$$(p, u, d) = (p_n, u_n, d_n).$$

Next time: how do we choose (p_n, u_n, d_n) to match the constraints above?

References I

Petters, Arlie O. and Xiaoying Dong (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*. Springer. ISBN: 978-1-4939-3783-7.