Department of Mathematics, University of Utah

Analysis of Numerical Methods I
MATH 6610 – Section 001 – Fall 2025
Homework 1
Norms and linear algebra

Due Wednesday, August 27, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Matrix subspaces) With $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices of appropriates sizes, prove the following:

(a) $\text{range}(\boldsymbol{AB}) \subseteq \text{range}(\boldsymbol{A})$.
(b) $\text{corange}(\boldsymbol{AB}) \subseteq \text{corange}(\boldsymbol{B})$.
(c) $\ker(\boldsymbol{AB}) \supseteq \ker(\boldsymbol{B})$.
(d) $\text{coker}(\boldsymbol{AB}) \supseteq \text{coker}(\boldsymbol{A})$.

2. (Inner and outer products) Let $\boldsymbol{A} \in \mathbb{R}^{K\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times L}$ be given matrices. The matrix product $\boldsymbol{AB} \in \mathbb{R}^{K\times L}$ has entries

$$(AB)_{k,\ell} = \sum_{q=1}^{n} (A)_{k,q}(B)_{q,\ell}, \qquad\qquad k = 1,\ldots,K,\ \ell = 1,\ldots,L.$$

Using only this definition, prove the following:

(a) (The product of two matrices is an array of inner products) If $\boldsymbol{R} \in \mathbb{R}^{K\times n}$ has rows $\{\boldsymbol{r}_k\}_{k=1}^{K}$ and $\boldsymbol{C} \in \mathbb{R}^{n\times L}$ has columns $\{\boldsymbol{c}_\ell\}_{\ell=1}^{L}$, then

$$(RC)_{k,\ell} = \boldsymbol{r}_k^T \boldsymbol{c}_\ell.$$

(b) (The product of two matrices is a sum of rank-1 matrices) If $\boldsymbol{R} \in \mathbb{R}^{n\times L}$ has rows $\{\boldsymbol{r}_j\}_{j=1}^{n}$ and $\boldsymbol{C} \in \mathbb{R}^{K\times n}$ has columns $\{\boldsymbol{c}_j\}_{j=1}^{n}$, then

$$\boldsymbol{CR} = \sum_{j=1}^{n} \boldsymbol{c}_j \boldsymbol{r}_j^T.$$

(c) What is the maximum possible rank of $\boldsymbol{CR}$ from part b? Justify your answer.

3. If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, and $\|\cdot\|$ is a(ny) norm on $\mathbb{C}^n$, provide necessary and sufficient conditions (with a proof) on $\boldsymbol{A}$ so that $\boldsymbol{x} \mapsto \|\boldsymbol{Ax}\|$ is a norm.

4. (Equivalence of ℓ^2 and ℓ^∞ norms) For vectors on $\mathbb{C}^n$, compute constants c, k in the norm equivalence statement

$$c\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq k\|\boldsymbol{x}\|_\infty, \quad \boldsymbol{x} \in \mathbb{C}^n,$$

and identify vectors $\boldsymbol{x} \in \mathbb{C}^n$ achieving these bounds.

5. (ℓ^1 and ℓ^∞ matrix induced norms) Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ be a matrix whose columns and rows are $\boldsymbol{c}_j$ and $\boldsymbol{r}_j$, respectively:

$$\boldsymbol{A} = \begin{pmatrix} | & & | \\ \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} — & \boldsymbol{r}_1^* & — \\ & \vdots & \\ — & \boldsymbol{r}_m^* & — \end{pmatrix}$$

With $\|\cdot\|_1$ and $\|\cdot\|_\infty$ the matrix norms induced by the vector ℓ^1 and ℓ^∞ norms, respectively, prove the following two statements:

$$\|\boldsymbol{A}\|_1 = \max_{j \in [n]} \|\boldsymbol{c}_j\|_1, \qquad \|\boldsymbol{A}\|_\infty = \max_{i \in [m]} \|\boldsymbol{r}_i\|_1.$$

6. (Norm submultiplicativity) A (matrix) norm $\|\cdot\|$ is said to be *submultiplicative* if $\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|$ for all matrices $\boldsymbol{A}, \boldsymbol{B}$ of conforming sizes.

(a) Prove that the matrix induced p-norm $\|\cdot\|_p$ is submultiplicative for any $p \in [1, \infty]$.

(b) Prove that the Frobenius norm $\|\cdot\|_F$ is submultiplicative.

(Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ can be rectangular above.)