Department of Mathematics, University of Utah

Analysis of Numerical Methods I

MATH 6610 – Section 001 – Fall 2025

Homework 6

Conditioning and LU

Due Wednesday, October 1, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Least squares problems) Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{C}^m$ be given. $\boldsymbol{A}^+$ denotes the Moore-Penrose pseudoinverse of $\boldsymbol{A}$.

(a) Assume $\mathrm{rank}(\boldsymbol{A}) = n$. Prove that $\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$ solves the least squares problem,

$$\operatorname*{argmin}_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2$$

(b) Show that when $\mathrm{rank}(\boldsymbol{A}) = n$, the least squares solution $\boldsymbol{x}$ equivalently is the unique solution to the *normal equations*, $\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$.

(c) For general $\boldsymbol{A}$, prove that $\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$ solves the minimum-norm least squares problem,

$$\min \|\boldsymbol{x}\|_2 \text{ subject to } \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2 \text{ is minimized}$$

2. (ϵ-pseudospectrum) Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$. We let $\Lambda(\boldsymbol{A})$ denote the spectrum (set of eigenvalues) of $\boldsymbol{A}$. For some $\epsilon > 0$, the ϵ-pseudospectrum of $\boldsymbol{A}$ is the set of complex numbers $\Lambda_\epsilon(\boldsymbol{A}) \subset \mathbb{C}$ given by

$$\Lambda_\epsilon(\boldsymbol{A}) \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; z \in \Lambda(\boldsymbol{A} + \Delta\boldsymbol{A}) \text{ for some } \Delta\boldsymbol{A} \in \mathbb{C}^{n\times n} \text{ satisyfing } \|\Delta\boldsymbol{A}\|_2 \leq \epsilon \right\}.$$

Another concept we'll need is the *resolvent*. The resolvent of $\boldsymbol{A}$ at $z \in \mathbb{C}$ is defined as the matrix $\mathrm{Res}_{\boldsymbol{A}}(z) \coloneqq (z\,\boldsymbol{I} - \boldsymbol{A})^{-1}$.

Given $\epsilon > 0$ and $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, prove that the following three sets of numbers in the complex plane are identical:

(a) $\Lambda_\epsilon(\boldsymbol{A})$

(b) The set of $z \in \mathbb{C}$ satisfying $\|(\boldsymbol{A} - z\boldsymbol{I})\boldsymbol{v}\|_2 \leq \epsilon$ for some unit vector $\boldsymbol{v}$.

(c) $\left\{ z \in \mathbb{C} \;\middle|\; \|\mathrm{Res}_{\boldsymbol{A}}(z)\|_2 \geq 1/\epsilon \right\}$, with the convention that $\|\mathrm{Res}_{\boldsymbol{A}}(z)\|_2 = \infty$ if $z \in \Lambda(\boldsymbol{A})$.

3. (Bauer-Fike eigenvalue perturbations) Suppose $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is diagonalizable with eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$. Let $\Delta\boldsymbol{A}$ be any matrix satisfying $\|\Delta\boldsymbol{A}\|_2 \leq \epsilon$. Suppose $\tilde{\lambda}$ is an(y) eigenvalue of $\boldsymbol{A} + \Delta\boldsymbol{A}$. Show that there is some eigenvalue $\lambda \in \Lambda(\boldsymbol{A})$ such that

$$\left|\lambda - \tilde{\lambda}\right| \leq \epsilon\,\kappa(\boldsymbol{V}),$$

with $\kappa(\boldsymbol{V})$ the (2-norm) condition number of $\boldsymbol{V}$. The typical way to prove this is to exploit the equivalence of sets defining the ϵ-pseudospectrum. (E.g., rewrite the resolvent with the knowledge that $\boldsymbol{A}$ is diagonalizable.). If $\boldsymbol{A}$ is unitarily diagonalizable, what does the bound above simplify to?

4. (LU with partial pivoting) Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be invertible. Prove that the row-pivoted LU decomposition algorithm always successfully computes $\boldsymbol{PA} = \boldsymbol{LU}$.

5. (Schur complements) Consider the block matrix

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix},$$

where $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\boldsymbol{D} \in \mathbb{C}^{n\times n}$, and $\boldsymbol{B}$ and $\boldsymbol{C}$ have appropriate (generally rectangular) size. Throughout this problem, assume that both $\boldsymbol{A}$ and $\boldsymbol{D}$ are invertible. This problem concerns, among other things, computing the solution vectors to a linear system involving $\boldsymbol{M}$. The *Schur complement* of the block $\boldsymbol{A}$ of the matrix $\boldsymbol{M}$ is defined as

$$\boldsymbol{M}/\boldsymbol{A} \coloneqq \boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \in \mathbb{C}^{n\times n},$$

Similarly, $\boldsymbol{M}/\boldsymbol{D} \coloneqq \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ is the Schur complement of $\boldsymbol{D}$ of the matrix $\boldsymbol{M}$. For this problem, you will be performing block matrix operations; in particular, block matrix multiplication works like matrix multiplication with scalars. Perform the following exercises:

(a) If $\boldsymbol{N}$ is a 2×2 matrix, then its LU decomposition can be determined by asserting the form,

$$\boldsymbol{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \boldsymbol{LU} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix},$$

and by performing matrix multiplication and solving for w, x, y, z, in terms of the entries of $\boldsymbol{N}$. Use this same idea to compute a block LU decomposition for $\boldsymbol{M}$ in terms of its blocks $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$.

(b) Show that $\det \boldsymbol{M} = (\det \boldsymbol{A})(\det(\boldsymbol{M}/\boldsymbol{A}))$, and hence that $\boldsymbol{M}$ is invertible iff both $\boldsymbol{A}$ and $\boldsymbol{M}/\boldsymbol{A}$ are invertible. (NB: the familiar formula for the determinant of a 2×2 matrix isn't true for block matrices...but you could compute a block LDU decomposition.)

(c) Assume $\boldsymbol{M}$ is Hermitian. Show that $\boldsymbol{M}$ is positive definite iff both $\boldsymbol{A}$ and $\boldsymbol{M}/\boldsymbol{A}$ are positive definite.

(d) For generally non-Hermitian $\boldsymbol{M}$, given vectors $\boldsymbol{f} \in \mathbb{C}^m$ and $\boldsymbol{g} \in \mathbb{C}^n$, consider the following linear system:

$$\boldsymbol{M}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}, \qquad \boldsymbol{x} \in \mathbb{C}^m, \;\; \boldsymbol{y} \in \mathbb{C}^n.$$

Using the block LU decomposition of $\boldsymbol{M}$, give a formula for $\boldsymbol{x}$ and $\boldsymbol{y}$ that utilizes inverses of only $\boldsymbol{A}$ and $\boldsymbol{M}/\boldsymbol{A}$.