Department of Mathematics, University of Utah

Analysis of Numerical Methods I
MATH 6610 – Section 001 – Fall 2025
Homework 7
The QR decomposition

Due Wednesday, October 22, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. (Hadamard's inequality) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ have columns $\boldsymbol{a}_j$ for $j \in [n]$. Prove that $|\det \boldsymbol{A}| \leq \prod_{j \in [n]} \|\boldsymbol{a}_j\|_2$.

2. (Cholesky and QR) Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ with $m \geq n$ be full rank, and have QR decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$. Write the Cholesky factor $\boldsymbol{L}$ of $\boldsymbol{A}^*\boldsymbol{A}$ in terms of $\boldsymbol{Q}$ and $\boldsymbol{R}$.

3. (Householder reflectors) Let $\boldsymbol{H} = \boldsymbol{I} - 2\boldsymbol{P}$ be a Householder reflector, where $\boldsymbol{P}$ is a rank-1 orthogonal projector. Show that $\boldsymbol{H}$ is Hermitian, unitary, and an involution (is its own inverse).

4. Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. Let $\boldsymbol{A}^k = \boldsymbol{Q}^{(k)}\boldsymbol{R}^{(k)}$ denote the QR decomposition of $\boldsymbol{A}^k$. ($\boldsymbol{A}^k$ is the k-fold product of $\boldsymbol{A}$, but the superscript in $\boldsymbol{Q}^{(k)}$ is just an index.) Consider the following iterative definition:

- $\boldsymbol{A}_1 = \boldsymbol{A}$
- $\boldsymbol{A}_j = \boldsymbol{Q}_j\boldsymbol{R}_j$ for $j \geq 1$ (the QR decomposition of $\boldsymbol{A}_j$).
- $\boldsymbol{A}_{j+1} \coloneqq \boldsymbol{R}_j\boldsymbol{Q}_j$ for $j \geq 1$.

Prove that for any $k \in \mathbb{N}$,

$$\boldsymbol{Q}^{(k)} = \prod_{j=1}^{k} \boldsymbol{Q}_j = \boldsymbol{Q}_1 \cdots \boldsymbol{Q}_k, \qquad \boldsymbol{R}^{(k)} = \prod_{j=k}^{1} \boldsymbol{R}_j = \boldsymbol{R}_k \cdots \boldsymbol{R}_1.$$

5. Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, and let $\boldsymbol{V}$ be a(ny) matrix whose columns are the eigenvectors of $\boldsymbol{A}$. (If $\boldsymbol{A}$ is non-defective, this is the standard eigenvector matrix, if $\boldsymbol{A}$ is defective then it's a matrix of generalized eigenvectors in the Jordan normal form of $\boldsymbol{A}$.) Consider the QR decomposition $\boldsymbol{V} = \boldsymbol{Q}\boldsymbol{R}$, and define $\boldsymbol{B} = \boldsymbol{Q}^*\boldsymbol{A}\boldsymbol{Q}$. Identify in general which entries of the matrix $\boldsymbol{B}$ are nonzero, and relate $(\boldsymbol{Q}, \boldsymbol{B})$ to another matrix decomposition we've discussed in class.