Department of Mathematics, University of Utah
Analysis of Numerical Methods I
MATH 6610 – Section 001 – Fall 2025
Homework 8
Power iteration

Due Wednesday, October 29, 2025

Submission instructions:
Submit your assignment on gradescope.

Problem assignment:

1. Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, with $p = \min\{m, n\}$, have column-pivoted QR decomposition given by $\boldsymbol{AP} = \boldsymbol{QR}$, where $\boldsymbol{P}$ is the permutation matrix, with the convention that the diagonal elements of $\boldsymbol{R}$ are chosen as non-negative numbers. Show that (a) $(R)_{j+1,j+1} \leq (R)_{j,j}$ for $j \in [p-1]$, and that (b)

$$\operatorname{rank}(\boldsymbol{A}) = \max\left\{ j \in [p] \;\middle|\; (R)_{j,j} > 0 \right\}.$$

2. (Column-pivoted QR as greedy determinant maximization) Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ have rank n, and suppose that the column-pivoted QR decomposition is applied to $\boldsymbol{A}$. In particular, the column pivoting strategy selects a sequence of column indices $\{c_1, \ldots, c_n\}$, such that $\boldsymbol{AP} = \boldsymbol{QR}$, where $\boldsymbol{P}$ is the permutation matrix satisfying $(\boldsymbol{P})_{c_j,j} = 1$. Show that, for $j \in [n-1]$,

$$c_{j+1} = \operatorname*{argmax}_{k \in [n]} \det(\boldsymbol{B}^* \boldsymbol{B}), \qquad \boldsymbol{B} = \boldsymbol{A}_{*C_{j,k}}, \qquad C_{j,k} = \{c_1, \ldots, c_j\} \bigcup \{k\},$$

where $\boldsymbol{A}_{*S}$ is the $m \times |S|$ matrix formed from the S-indexed columns of $\boldsymbol{A}$.

3. Let $\boldsymbol{x} \in \mathbb{C}^n$ be fixed but arbitrary, and nonzero. The Householder reflector $\boldsymbol{H}$ taking $\boldsymbol{x}$ to a multiple of $\boldsymbol{e}_1$ has the form $\boldsymbol{H} = \boldsymbol{I} - 2\boldsymbol{P} = \boldsymbol{I} - 2\boldsymbol{v}\boldsymbol{v}^*$, for some unit norm vector $\boldsymbol{v} \in \mathbb{C}^n$. Show that

$$\boldsymbol{v} = \frac{\boldsymbol{x} - c\boldsymbol{e}_1}{\|\boldsymbol{x} - c\boldsymbol{e}_1\|_2}, \qquad \frac{c}{\|\boldsymbol{x}\|_2} = \begin{cases} e^{i\theta}, & x_1 = 0, \text{ with } \theta \in \mathbb{R} \text{ arbitrary} \\ \pm\frac{x_1}{|x_1|}, & x_1 \neq 0. \end{cases}$$

4. (Hessenberg matrices) A matrix is an *upper Hessenberg* matrix if its entries below its main subdiagonal vanish. I.e., $\boldsymbol{H}$ is upper Hessenberg if $(H)_{j,k} = 0$ for $j > k + 1$. Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$.

(a) Use (a sequence of) Householder reflectors to construct a unitary similarity transform that takes $\boldsymbol{A}$ to an upper Hessenberg matrix. (And hence preserves its spectrum.)

(b) Assume $\boldsymbol{A}$ is Hermitian. What does this imply about the structure of the transformed version of $\boldsymbol{A}$ from the previous part? In this context, why might this transformation be useful for (numerically) determining the spectrum of $\boldsymbol{A}$?

5. (The Rayleigh-Ritz method) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, and let $\boldsymbol{U} \in \mathbb{C}^{n \times p}$ with $p \leq n$ have orthonormal columns. Consider $\boldsymbol{B} = \boldsymbol{U}^* \boldsymbol{A} \boldsymbol{U} \in \mathbb{C}^{p \times p}$. If $(\lambda, \boldsymbol{v})$ is a(ny) eigenpair of $\boldsymbol{A}$, show that λ is an eigenvalue of $\boldsymbol{B}$ if $\boldsymbol{v} \in \operatorname{range}(\boldsymbol{U})$, and describe how to compute the eigenvector $\boldsymbol{v}$ from the $\boldsymbol{U}$ and the corresponding eigenpair of $\boldsymbol{B}$.

6. (Arnoldi iteration) Let $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and let $\boldsymbol{q}_1 \in \mathbb{C}^n$ be an arbitrarily chosen unit-norm vector. The *Arnoldi iteration* is the implementation of (modified) Gram-Schmidt orthogonalization to the sequence of vectors $\boldsymbol{q}_1$, $\boldsymbol{A}\boldsymbol{q}_1$, $\boldsymbol{A}^2\boldsymbol{q}_1$, ..., $\boldsymbol{A}^{k-1}\boldsymbol{q}_1$ for some $k \leq n$, which produces a sequence of orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, ..., $\boldsymbol{q}_k$. For this problem, you may assume that these k vectors are linearly independent.

(a) Show that the Arnoldi iteration equivalently can be implemented iteratively, by first orthogonalizing $\boldsymbol{A}\boldsymbol{q}_1$ against $\boldsymbol{q}_1$ to produce $\boldsymbol{q}_2$, then orthogonalizing $\boldsymbol{A}\boldsymbol{q}_2$ against $\boldsymbol{q}_1, \boldsymbol{q}_2$ to produce $\boldsymbol{q}_3$, then orthogonalizing $\boldsymbol{A}\boldsymbol{q}_3$ against $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ to produce $\boldsymbol{q}_4$, etc.
(In fact this implementation is "the" Arnolid iteration, rather than the previously described approach, and the point here is that one never directly computes the action of $\boldsymbol{A}^k$ for large k against a vector.)

(b) Fix $p \leq n$, and let $\boldsymbol{Q} \in \mathbb{C}^{n \times p}$ be a matrix whose columns are $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p$ from the Arnoldi iteration ($p \leq k$). Define $\boldsymbol{B} \coloneqq \boldsymbol{Q}^*\boldsymbol{A}\boldsymbol{Q}$. What can you say about the structure/entries of $\boldsymbol{B}$?

(c) Informally, how would you expect the spectrum of $\boldsymbol{B}$ to relate to that of $\boldsymbol{A}$?
(I'm not asking for a proof here, but just an educated and well-motivated prediction.)