Math 6610: Analysis of Numerical Methods, I
Linear algebraic preliminaries

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Accompanying text: Trefethen and Bau 1997, Lectures 1, 2, 3
Atkinson 1989, Sections 7.1, 7.3
Salgado and Wise 2022, Sections 1.1, 1.2

Notation

We'll use some standard math notation

- $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$
- $\in$, $\forall$, $\exists$, !
- $\left\{x \in \mathbb{C} \;\middle|\; \mathrm{Im}\{x\} \in \mathbb{N}\right\}$
- $z = x + iy$ for $x, y \in \mathbb{R} \Longrightarrow \bar{z} = z^* := x - iy$

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Vectors, matrices, etc:

- $\boldsymbol{u} \in \mathbb{C}^n$
- $\boldsymbol{A} \in \mathbb{C}^{m \times n}$
- linear independence
- rank
- (conjugate) transpose
- determinant
- matrix inverse
- subspaces defined by $\boldsymbol{A}$: range, kernel, cokernel, corange

Hilbertian structure

$\mathbb{C}^n$ endowed with the standard inner product is a Hilbert space. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n$,

- $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, $\|\boldsymbol{u}\|$
- $\angle(\boldsymbol{u}, \boldsymbol{v})$
- $\boldsymbol{u} \perp \boldsymbol{v}$
- $\mathrm{Proj}_{\boldsymbol{v}} \boldsymbol{u}$
- orthogonal and orthonormal sets

All the above is also well-defined in $\mathbb{R}^n$.

Matrices and matrix algebra

An $m \times n$ matrix $\boldsymbol{A}$ is a tableau of elements (from $\mathbb{R}$ or $\mathbb{C}$):

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m \times n}, \qquad (A)_{j,k} = a_{j,k}, \qquad j \in [m], k \in [n]$$

A matrix is "just" a vector with "2D" indices.

Matrices come with a natural algebra, i.e., sum and product operations involving matrices:

- Product of a scalar and a matrix
- Sum of two matrices (of the same size)
- Product of two matrices (of conforming sizes)

$$\boldsymbol{A} \in \mathbb{C}^{m \times n}, \boldsymbol{B} \in \mathbb{C}^{n \times k} \quad \Longrightarrow \quad \boldsymbol{AB} \in \mathbb{C}^{m \times k}, \;\; (AB)_{j,k} = \sum_{\ell=1}^{n} a_{j\ell} b_{\ell k}.$$

The "four fundamental subspaces"

Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ be given. The four fundamental subspaces are uniquely defined:

- $\mathbb{C}^m \supset \mathcal{R}(\boldsymbol{A}) = \mathrm{range}(\boldsymbol{A}) = \mathrm{Im}(\boldsymbol{A})$, the "column space" of $\boldsymbol{A}$
- $\mathbb{C}^n \supset \mathcal{R}(\boldsymbol{A}^*) = \mathrm{corange}(\boldsymbol{A})$, the "row space" or "corange" of $\boldsymbol{A}$.
- $\mathbb{C}^n \supset \mathcal{K}(\boldsymbol{A}) = \mathrm{ker}(\boldsymbol{A})$, the "nullspace" or "kernel" of $\boldsymbol{A}$
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Essentially by definition: $\mathcal{K}(\boldsymbol{A})$ contains all vectors $\boldsymbol{v}$ satisfying $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$. I.e., if $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathbb{C}^n$ are conjugate-tranposed rows of $\boldsymbol{A}$, then $\boldsymbol{v}$ is orthogonal to $\mathrm{span}\{\boldsymbol{w}_1, \ldots \boldsymbol{w}_n\}$.

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Theorem (Fundamental Theorem of Linear Algebra)

For any $\boldsymbol{A} \in \mathbb{C}^{m \times n}$,

$$n = \dim \mathrm{corange}(\boldsymbol{A}) + \dim \ker(\boldsymbol{A}), \qquad \mathrm{corange}(\boldsymbol{A}) \perp \ker(\boldsymbol{A}), \qquad \mathbb{C}^n = \mathrm{corange}(\boldsymbol{A}) \oplus \ker(\boldsymbol{A})$$
$$m = \dim \mathrm{range}(\boldsymbol{A}) + \dim \mathrm{coker}(\boldsymbol{A}), \qquad \mathrm{range}(\boldsymbol{A}) \perp \mathrm{coker}(\boldsymbol{A}), \qquad \mathbb{C}^m = \mathrm{range}(\boldsymbol{A}) \oplus \mathrm{coker}(\boldsymbol{A})$$

Norms

Metrizing linear spaces is a big business in mathematics.

Given a vector space V, a map $\|\cdot\| : V \to \mathbb{R}$ is a *norm* if it satisfies all the following properties:

- $\|\boldsymbol{x}\| \geqslant 0 \;\; \forall \boldsymbol{x} \in V$
- $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \;\; \forall \boldsymbol{x}, \boldsymbol{y} \in V$
- $\|c\boldsymbol{x}\| = |c|\|\boldsymbol{x}\| \;\; \forall \boldsymbol{x} \in V, \; c \in \mathbb{C}$.

We are mostly concerned with standard examples $V = \mathbb{R}^n, \mathbb{C}^n, \mathbb{C}^{m \times n}$, etc.

Vector norms

The "standard" examples of vector norms are the ℓ^p norms.

With $\boldsymbol{x} \in \mathbb{C}^n$:

$$\|\boldsymbol{x}\|_p^p := \sum_{j \in [n]} |x_j|^p, \qquad p \in [1, \infty)$$

$$\|\boldsymbol{x}\|_\infty := \max_{j \in [n]} |x_j|, \qquad p = \infty.$$

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Example

Show that $\|\cdot\|_2$ on $\mathbb{C}^n$ is a norm.

Matrix norms

One straightforward identification of norms on matrices are "entrywise" ones:

$$\|\boldsymbol{A}\|_{p,p} := \|\mathrm{vec}(\boldsymbol{A})\|_p, \qquad\qquad p \in [1, \infty],$$

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There are "mixed" entrywise norm definitions, corresponding to taking ℓ^p vector norms of each row, and then a vector ℓ^q norm of the resulting vector of norms,

$$\|\boldsymbol{A}\|_{p,q} := \left(\sum_{j\in[n]} \left(\sum_{i\in[m]} |a_{i,j}|^p \right)^{q/p} \right)^{1/q}.$$

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A particularly useful entrywise norm is the *Frobenius norm*,

$$\|\boldsymbol{A}\|_F := \|\boldsymbol{A}\|_{2,2}.$$

Induced matrix norms

A more conceptual collection of matrix norms are *induced* by vector norms.

By viewing $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ as the mapping $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$, norms can be defined as the maximum relative "size" of this mapping:

$$\|\boldsymbol{A}\|_p := \sup_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}, \qquad\qquad p \in [1, \infty].$$

(Note that $\boldsymbol{A}$ can be rectangular here.)

That these are proper norms is direct from the fact that $\|\cdot\|_p$ is a norm on $\mathbb{C}^m$.

Norm equivalence

A rather important and useful fact is that any two norms on the same finite-dimensional vector space are *equivalent*.

Theorem (All norms on a finite-dimensional space are equivalent)

Let V be an n-dimensional vector space, and let $\|\cdot\|_*$ and $\|\cdot\|_+$ be any two norms on this space. Then there are strictly positive constants c and k such that for all $\boldsymbol{x} \in V$,

$$c\|\boldsymbol{x}\|_* \leqslant \|\boldsymbol{x}\|_+ \leqslant k\|\boldsymbol{x}\|_*.$$

The constants c and k can depend on V (in particular n) and the choice of $\|\cdot\|_*$ and $\|\cdot\|_+$, but not on $\boldsymbol{x}$.

Note that the above applies equally to spaces containing vectors or matrices.

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Note that the above applies equally to spaces containing vectors or matrices.

The good news: Norm equivalence suggests it doesn't matter which norm you pick.

The bad news: To prove something, it typically matters which norm you pick.

Example

Compute c and k such that,

$$c\|\boldsymbol{x}\|_1 \leqslant \|\boldsymbol{x}\|_2 \leqslant k\|\boldsymbol{x}\|_1, \qquad \forall\, \boldsymbol{x} \in \mathbb{C}^n.$$

Also, identify examples of vectors $\boldsymbol{x}$ that achieve the upper and lower bounds above.

Example

Compute c and k such that,

$$c\|\boldsymbol{A}\|_1 \leqslant \|\boldsymbol{A}\|_2 \leqslant k\|\boldsymbol{A}\|_1, \qquad \forall\ \boldsymbol{A} \in \mathbb{C}^{m\times n}$$

Also, identify examples of matrices $\boldsymbol{A}$ that achieve the upper and lower bounds above.

Example

Compute $\|\boldsymbol{A}\|_2$, where,

$$\boldsymbol{A} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}$$

Hilbertian structure

Of special interest are the norms arising from inner products: these norms induce Euclidean-like geometry (Hilbert spaces).

The prototypical example on $\mathbb{C}^n$ is the ℓ^2 norm: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, we have,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^* \boldsymbol{x} = \sum_{i \in [n]} x_i y_i^*, \qquad\qquad \|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle .$$

Inner products are *bilinear* forms (technically "sesquilinear" for the complex field $\mathbb{C}$).

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One of the most useful algebraic properties of inner products that give rise to a norm $\| \cdot \|$ is the *Cauchy-Schwarz* inequality:

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leqslant \|\boldsymbol{x}\| \|\boldsymbol{y}\|$$

From this property one can observe that the following geometric structure of elements $\boldsymbol{x}, \boldsymbol{y}$ is reasonable:

$$\cos\left(\angle(\boldsymbol{x}, \boldsymbol{y})\right) := \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|}, \qquad \boldsymbol{x} \perp \boldsymbol{y} \text{ iff } \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0.$$

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There are Hilbertian norms even for matrices, with a common example being the Frobenius norm:

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_F := \operatorname{Tr}\left(\boldsymbol{B}^* \boldsymbol{A}\right), \qquad \|\boldsymbol{A}\|_F^2 = \langle \boldsymbol{A}, \boldsymbol{A} \rangle_F$$

The Pythagorean Theorem

The sledgehammer killing an ant way to prove the Pythagorean Theorem: Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ be two orthogonal vectors (say in $\mathbb{C}^n$).

Since $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = 0$, then

$$\begin{aligned} \|\boldsymbol{x}_1 + \boldsymbol{x}_2\|_2^2 &= \langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{x}_1 + \boldsymbol{x}_2 \rangle \\ &= \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle + \langle \boldsymbol{x}_2, \boldsymbol{x}_2 \rangle + \underbrace{\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle}_{0} + \underbrace{\langle \boldsymbol{x}_2, \boldsymbol{x}_1 \rangle}_{0} \\ &= \|\boldsymbol{x}_1\|_2^2 + \|\boldsymbol{x}_2\|_2^2 . \end{aligned}$$

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One of the more useful extensions of this (not apparent from $n = 2$) is: If $\boldsymbol{x}_1, \ldots \boldsymbol{x}_k$ are k mutually orthogonal vectors in $\mathbb{C}^n$, then,

$$\left\| \sum_{j \in [k]} \boldsymbol{x}_j \right\|_2^2 = \sum_{j \in [k]} \|\boldsymbol{x}_j\|_2^2$$

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