Math 6610: Analysis of Numerical Methods, I
Approximation with Fourier Series

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Resources: Hesthaven, S. Gottlieb, and D. Gottlieb 2007, Chapters 1-2
Canuto et al. 2011, Chapters 2.1, 5.1
Shen, Tang, and Wang 2011, Chapter 2

Function approximation

The conceptual goal of approximation is to transform objects from a high-dimensional space (ideally an infinite-dimensional one), to a low-dimensional *encoding*.
One main reason is for ease of computational manipulation + approximation.

Loosely speaking, we wish gather some data of a function, e.g., points in space, and "connect the dots".

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There are two scenarios we'll consider:

- approximation with Fourier Series
- approximation with polynomials

We'll start with the first case, which is more transparent in some ways than the other case.

These examples both have essentially the same message, which is:

$$\textbf{Smoothness} \Longrightarrow \textbf{Compressibility}$$

Fourier Series, I

A simple example of a global approximation scheme is a Fourier Series.

Consider a given $u : [0, 2\pi] \to \mathbb{C}$, which we represent as a sum of complex exponentials,

$$u(x) \approx \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x), \qquad \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

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One straightforward strategy to identify $\widehat{u}_k$ is to choose them to minimize a loss,

$$\widehat{u}_k = \operatorname*{arg\,min}_{\widehat{u}_k, k \in \mathbb{Z}} \left\| u(x) - \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x) \right\|_2^2,$$

where we have introduced the norm and a corresponding inner product,

$$\langle f, g \rangle := \int_0^{2\pi} f(x) g(x)^* \mathrm{d}x, \qquad \|f\|_2^2 := \langle f, f \rangle,$$

where z^* is the complex conjugate of z. [1]

[1] We are mostly interested in real-valued functions, so the introduction of complex arithmetic is somewhat artificial here. We could write the basis as real-valued $\sin kx$ and $\cos kx$ functions with real coefficients. This achieves the same results but uses somewhat more technical formulas.

We have conveniently chosen ϕ_k as an orthonormal basis, so that,

$$\langle \phi_k, \phi_\ell \rangle = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

There is a unique solution for the $\widehat{u}_k$ that minimizes the loss, and using basis orthonormality the solution has a fairly simple expression,

$$\widehat{u}_k = \langle u, \phi_k \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x.$$

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This gives us a first taste of some functional analysis: Define,

$$L^2 = L^2\left([0, 2\pi]; \mathbb{C}\right) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \|f\|_2^2 < \infty \right\}.$$

Fourier Series representations are complete in L^2:

$$u \in L^2 \quad \Longrightarrow \quad \lim_{N \to \infty} \left\| u(x) - \sum_{k=-N}^{N} \widehat{u}_k \phi_k(x) \right\|_2 = 0,$$

and orthonormality of the basis results in Parseval's identity,

$$u \in L^2 \quad \Longrightarrow \quad \|u\|_2^2 = \sum_{k \in \mathbb{Z}} \left| \widehat{u}_k \right|^2.$$

Fourier approximation

$$u(x) \overset{L^2}{=} \sum_{k\in\mathbb{Z}} \widehat{u}_k \phi_k(x), \qquad \widehat{u}_k = \langle u, \phi_k \rangle$$

This is all well and good, but how does this serve us *computationally*?

With finite storage, we have to truncate the infinite series,

$$u(x) \approx u_N(x) := \sum_{|k|\leqslant N} \widehat{u}_k \phi_k(x)$$

How well does u_N approximate u?

Fourier compressibiilty

So our question regards how *compressible* the infinite series is with respect to the truncation N:

$$\|u - u_N\|_2 \overset{?}{\lesssim} h(N),$$

for some function $h(N)$.

- h decays quickly with $N \to u$ is very compressible
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There is, of course, the other pesky issue that in practice we cannot actually compute $\widehat{u}_k$ or u_N exactly, and so must resort to additional approximations.

But let's focus on one sin at a time....

Projections

Before investigating Fourier approximation results, it's worthwhile to introduce additional concepts: Projections.

Given an operator $P : L^2 \to V$, where $V \subset L^2$ is some subspace of L^2, then P is a projection operator if

$$P^2 = P.$$

The action $u \mapsto Pu$ projects u onto V.
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A projection operator P is orthogonal if $W \perp V$, equivalently if for every $u, v \in L^2$:

$$P = P^*,$$

where P^* is the *adjoint* of P, defined as the operator P^* satisfying

$$\langle Pu, v\rangle := \langle u, P^* v\rangle.$$

(Notice that $P = P^2$ being a projector and also $P = P^*$ being an orthogonal projector matches our linear algebraic notation/definition....)

Truncation and projection

We are considering the truncation,

$$\sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x) \stackrel{L^2}{=} u \approx u_N = \sum_{|k| \leqslant N} \widehat{u}_k \phi_k(x).$$

This truncation is an orthogonal projector.

Theorem

Define P_N as the operator,

$$P_N u = u_N = \sum_{|k| \leqslant N} \widehat{u}_k \phi_k(x), \qquad u \stackrel{L^2}{=} \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k.$$

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To really hammer the point home in terms of familiar linear algebra concepts:

- $\mathrm{range}(P_N) = \mathrm{span}\{e^{ikx}\}_{|k| \leqslant N}$
- $\ker(P_N) = \mathrm{span}\{e^{ikx}\}_{|k| > N}$
- $\mathrm{range}(P_N) \perp \ker(P_N)$, and $\mathrm{range}(P_N) \oplus \ker(P_N) = L^2$

A basic approximation estimate, I

Can we bound $\|u - P_N u\|_2$? First note that,

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As long as u is differentiable and $k \neq 0$, integration by parts is our friend:

$$\begin{aligned}
\widehat{u}_k &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x \\
&= \frac{i}{k\sqrt{2\pi}} u(x) e^{-ikx} \Big|_0^{2\pi} - \frac{i}{k\sqrt{2\pi}} \int_0^{2\pi} u'(x) e^{-ikx} \mathrm{d}x.
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Note that, conveniently, the first term vanishes if $u(0) = u(2\pi)$.
This is, of course, quite reasonable since we are approximating with periodic functions.

Note also that the remaining integral is the Fourier series coefficient for the derivative, $u'(x)$:

$$u'(x) = \sum_{|k| \in \mathbb{Z}} \widehat{u'}_k \phi_k(x), \qquad \widehat{u'}_k = \left\langle u', \phi_k \right\rangle .$$

and this makes sense so long as $u' \in L^2$.

A basic approximation estimate, II

Thus, if u is periodic and $u' \in L^2$ (so that $\widehat{u'}_k$ is well-defined), then

$$\widehat{u}_k = -\frac{i}{k}\widehat{u'}_k. \quad \|u - P_N u\|_2^2 \qquad = \sum_{|k|>N} |\widehat{u}_k|^2,$$

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This reveals a basic estimate for Fourier series coefficients:

$$\begin{aligned}\|u - P_N u\|_2^2 = \sum_{|k|>N} \frac{1}{|k|^2}\left|\widehat{u'}_k\right|^2 \leqslant \frac{1}{N^2}\sum_{|k|>N}\left|\widehat{u'}_k\right|^2 &\leqslant \frac{1}{N^2}\sum_{k\in\mathbb{Z}}\left|\widehat{u'}_k\right|^2 \\ &= \frac{1}{N^2}\left\|u'\right\|_2^2,\end{aligned}$$

where the last relation is Parseval's identity.

A basic approximation estimate, II

We have just proven the following:

Theorem

Suppose $u, u' \in L^2$, *and that* $u(0) = u(2\pi)$. *Then,*

$$\|u - P_N u\|_2 \leqslant \frac{1}{N} \|u'\|_2$$

Sobolev spaces

To generalize this result, some additional notation will be helpful.

Definition (Sobolev spaces)

Given $s \in \mathbb{N}_0 = \{0, 1, \ldots, \}$, the ($L^2$ periodic) Sobolev space of functions is given by,

$$H_p^s([0, 2\pi]; \mathbb{C}) := \Big\{ f : [0, 2\pi] \to \mathbb{C} \;\Big|\; f^{(k)} \in L^2([0, 2\pi]; \mathbb{C}) \text{ for all } 0 \leqslant k \leqslant s, \\ f^{(k)}(0) = f^{(k)}(2\pi) \text{ for all } 0 \leqslant k \leqslant s - 1 \Big\}$$

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The *norm* on H_p^s is defined as,

$$\|u\|_{H^s}^2 := \sum_{k=0}^{s} \left\| u^{(k)} \right\|_2^2 .$$

Some specializations of interest:

- $s = 0 \Longrightarrow H^0 = L^2$
- $s > 0 \Longrightarrow$ continuous functions $\subset H_p^s$

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The parameter s encodes the "amount" of smoothness that functions have, with inclusions:

$$H_p^r \subset H_p^s, \qquad r > s \geqslant 0.$$

General approximation results

The language of Sobolev spaces is the standard language in which to technically describe convergence rates of Fourier Series approximations. (Note below that $s = 1$ recovers our previous result.)

Theorem

If $u \in H^s$, then

$$\|u - P_N u\|_2 \leqslant N^{-s} \|u\|_{H^s}$$

Related to degrees of freedom, $M = 2N + 1$, then $\|u - P_N u\|_{L^2} \lesssim M^{-s} \|u\|_{H_p^s}$.

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Actually, something even stronger is true about Fourier approximation:

Theorem

If $u \in H^s$, then for every $0 \leqslant r < s$,

$$\|u - P_N u\|_{H_p^r} \leqslant N^{-(s-r)} \|u\|_{H_p^s}.$$

This result demonstrates tradeoff between smoothness of the function versus the strength of the norm under which convergence is sought.

Infinite regularity

The previous Fourier series results are essentially "good enough" to understand the basic point that smoothness of u translates into efficient approximations.
But to close the loop: what if u is infinitely differentiable?

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Theorem

Let $u : [0, 2\pi) \to \mathbb{C}$ be the restriction of a function $f : \mathbb{C} \to \mathbb{C}$ to the unit circle. I.e., $u(x) := f(e^{ix})$. Assume f is (complex) analytic in an annular neighborhood of the unit circle in $\mathbb{C}$. (This implies that u is infinitely differentiable.)

Then there exist constants $K, c > 0$, and fixing $s \in \mathbb{N}_0$ there are there are constants $\widetilde{K}, \widetilde{c} > 0$ such that,

$$\|u - P_N u\|_2 \leqslant K e^{-cN}, \qquad \|u - P_N u\|_{H_p^s} \leqslant \widetilde{K} e^{-\widetilde{c}N}.$$

The constants $K, c, \widetilde{K}, \widetilde{c}$ depend on the radii defining the annular region of analyticity.

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Proof steps:

- $P_N u$ is a truncated Laurent series of f around the origin in $\mathbb{C}$.
- Convergence of the Laurent series in the region $r_1 \leqslant |z| \leqslant r_2$, where $r_1 < 1 < r_2$, can be used to estimate the truncated Laurent series coefficients.

The overall theme

The results we've described are *generic* lessons, even for nonperiodic global approximation:

- Such global methods have (rates of) accuracy that are limited only by functional regularity
  - Finite regularity $\Longrightarrow$ polynomial rates of error decay
  - Infinite regularity $\Longrightarrow$ superpolynomial (often exponential) rates of error decay
    (Note that real analyticity is not sufficient for complex analyticity; lack of complex analyticity generally downgrades pure exponential convergence to subexponential.)
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    (Note that real analyticity is not sufficient for complex analyticity; lack of complex analyticity generally downgrades pure exponential convergence to subexponential.)
- Global discretization methods are typically called *spectral methods*.
- The very succinct punchline:

Smoothness $\Longrightarrow$ Compressibility

References I

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