Math 1210: Calculus I
Introduction to limits

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Spring 2025

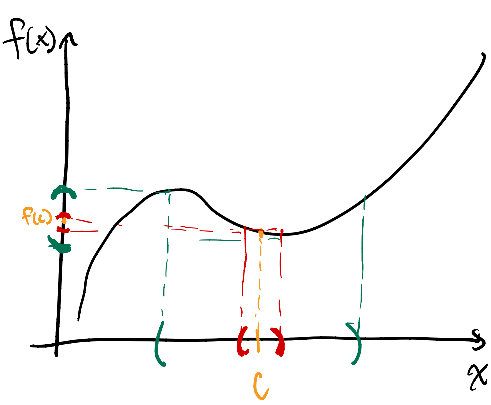
Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.1

Limits

Calculus is the study of *limits*.

Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

Pictorally:



As we shrink a window around $x = c$, the values that f takes over that window get closer to $f(c)$.

Limits

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Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

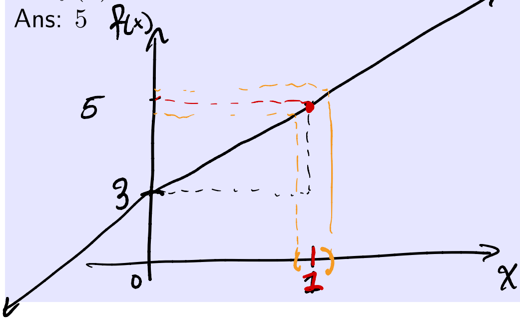
For many functions, the "answer" to this question is relatively boring:

Example

slope y-int.

Let $f(x) = 2x + 3$. What should the "limit" of $f(x)$ be as x gets close to 1?

Ans: 5



- when x is close to 1, $f(x)$ is close to 5.
- the closer x is to 1, the closer $f(x)$ is to 5.
- The "limit" of $f(x)$ as x goes to 1 is 5.

Aside: What if you didn't know how to plot $f(x)$?

We can tabulate values to get a sense for the limit:

x	$f(x) = 2x+3$
2	7
1.5	6
1.25	5.5
1.1	5.2
1.01	5.02
1.001	5.002

$\downarrow$

5

$\uparrow$

	4.998
0.999	4.98
0.99	4.8
0.9	4.5
0.75	4
0.5	3
0	

Limits

Calculus is the study of *limits*.

Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

The task of computing limits seems somewhat contrived, but is *extremely* useful in various practical scientific scenarios.

The previous examples were (hopefully) somewhat transparent.

The interesting cases appear when its not clear what value a limit takes

Why limits?

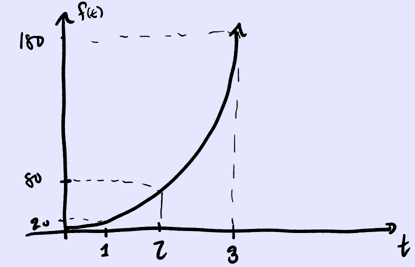
Here are a couple of examples about why limits are useful.

Example

Suppose a car's position along a straight road as a function of time is described by the function $f(t)$, where t is time.

Limits are useful for computing *velocity*.

Suppose $f(t) = 20 \cdot t^2$, where $f(t)$ measured in miles & t in hours



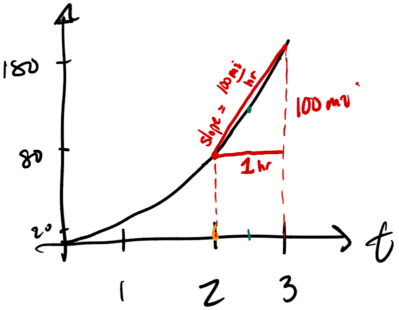
$f(3) = 180$ means you've gone 180 miles in 3 hours.

Your average speed in those 3 hours is

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{180 \text{ mi}}{3 \text{ hr}} = 60 \frac{\text{mi}}{\text{hr}}$$

Clearly, your speed is not constant. In the first hour, you go 20 miles, in the second hour you go (an additional) 60 miles, and in the 3rd hour, you go an additional 100 miles.

Question: How fast were you moving at $t=2$?



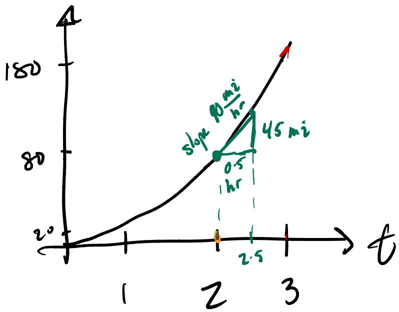
First Attempt:

$$(\text{avg})\ \text{speed} = \frac{\text{distance}}{\text{time}}$$

Measured between $t=2$, $t=3$

$$\frac{f(3)-f(2)}{1} = \frac{180-80}{1} = 100$$

The average speed in this interval is $100\ \frac{mi}{hr}$



Better:

avg speed between $t=2$ and $t=2.5$

$$\frac{f(2.5)-f(2)}{0.5} = \frac{125-80}{0.5} = \frac{45}{0.5} = 90$$

The average speed in this interval is $90\ \frac{mi}{hr}$

Best: Define the "instantaneous velocity" at $t=0$ to be the limit at $s \to 0$ of the average velocity

$$\frac{f(2+s)-f(2)}{s}$$

in the interval $(2, 2+s)$. We can compute this!

For any $s \neq 0$,

$$\frac{f(2+s)-f(2)}{s} = \frac{20(2+s)^2 - 20(2)^2}{s} = \frac{20}{s}\left[(2+s)(2+s) - 4\right]$$

$$= \frac{20}{s}\left[4+4s+s^2-4\right] = \frac{20}{s}\left[4s+s^2\right] = 80 + 20s$$

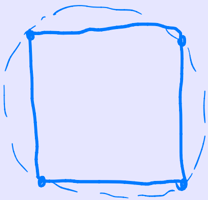
As $s \to 0$, $80 + 20s \longrightarrow 80$. So the speed at $t=2$ is $80\ \frac{mi}{hr}$

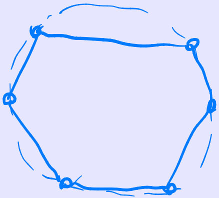
Why limits?

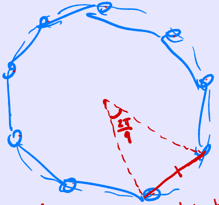
Here are a couple of examples about why limits are useful.

Example

We know the circumference of a circle of radius r (it's $2\pi r$).
If we didn't know this formula, we could compute the circumference of a circle by inscribing a polygon, computing the exact perimeter of this polygon, and using limits.







Note: can use trig to compute lengths of the edge

more vertices $\Rightarrow$ more accuracy

What happens w/ ∞'ly many vertices?

Idea dates back ≈ 2200 years to Archimedes (that ancient Greek mathematician who took a bath with a crown)

He called it the "method of exhaustion"

We'll use the notation,

$$\lim_{x \to c} f(x),$$

to denote "the limit of f as x approaches c".

For now, c should be some (finite) real number.

Before we saw that as $x \to 1$, the limit of $f(x) = 2x+3$ is 5.

We write this: $\lim_{x \to 1} (2x+3) = 5$.

Limit concepts and notation

We'll use the notation,

$$\lim_{x \to c} f(x),$$

to denote "the limit of f as x approaches c".

For now, c should be some (finite) real number.

If the value of this limit is some number L (another finite real number), that means that whenever x is close but not equal to c, then the value of $f(x)$ must be close to L.

The value $f(c)$ need not be equal to L, and $f(c)$ need not even be defined!

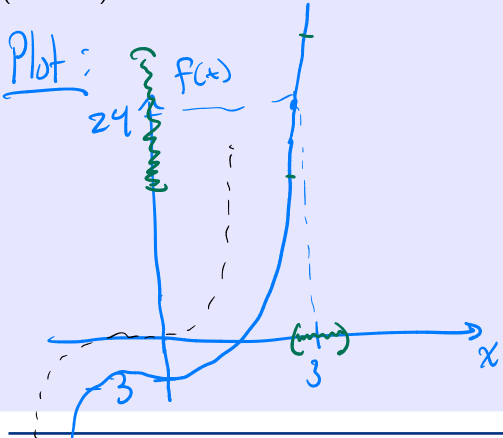
The limit $\lim_{x \to c} f(x)$ cannot have more than one value. Either it has a unique value, or it "does not exist".

note: we don't actually care what the value at 3 is

Example

Compute $\lim_{x\to 3}(x^3 - 3)$

(Ans: 24)



A quick plot suggests that both $f(3) = 24$ and "near" $x=3$, $f(x)$ is "near" 24.

Thus,

$$\lim_{x\to 3}(x^3-3) = 24$$

A nontrivial example

Example

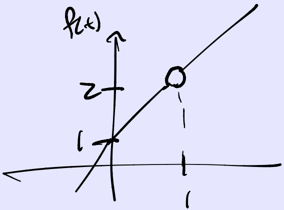
Compute $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

(Ans: 2)

First, note that $\frac{(x^2-1)}{x-1} = \frac{(x+1)(x-1)}{(x-1)}$

If $x \neq 1$, then we can divide the numerator & denominator by $(x-1)$ so that

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



We can see from the graph that when x is close to 1 but not equal to 1, $f(x)$ is close to 2.

Thus, $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$ even though f not defined @ $x=1$

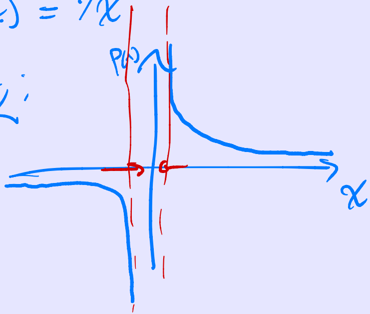
Limits need not exist

Example

Compute $\lim_{x \to 0} \frac{1}{x}$
(Ans: does not exist)

$f(x) = 1/x$

Plot:



In a small window around $x=0$, f doesn't stay near any value.

If $x > 0$ is small, then $f(x)$ is large & positive

If $x < 0$ is small, the $f(x)$ is large & negative

We say the limit does not exist (which is fine!)

A harder example

Example

Compute $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

(Ans: 1)

Let $f(x) = \dfrac{\sin x}{x}$.

How to plot?

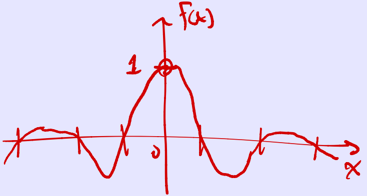
no idea?

options

1.) Tabulate values

2.) or, use graphing calculator

would find



It appears

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

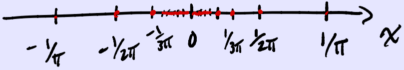
Even though $f(x)$ is not defined at $x=0$

Example

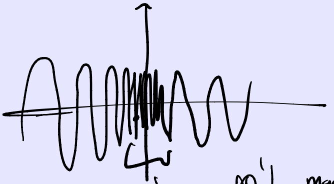
Compute $\lim\limits_{x\to 0} \sin \dfrac{1}{x}$ (Ans: does not exist)

Recall: $\sin\theta = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

so, $\sin\left(\frac{1}{x}\right) = 0$ when $x = \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi} \ldots$



f is zero at all these points and non zero in between so it __can't__ approach any limiting value as $x \to 0$



∞'ly many oscillations.

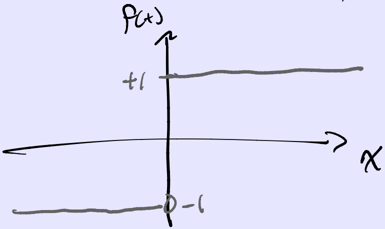
We write $\lim\limits_{x\to 0} \sin(1/x)$ D.N.E.

Jump discontinuities

Example

Compute $\lim_{x \to 0} \frac{x}{|x|}$ (Ans: does not exist)

Recall $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \Rightarrow f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0 \end{cases}$



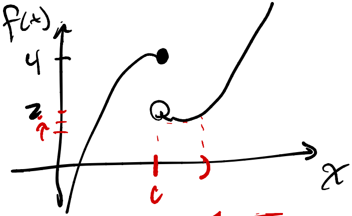
$\lim_{x \to 0} \frac{x}{|x|}$ does not exist

One-sided limits

The last example motivates an opportunity to consider "one-sided" limits.

We'll define $\lim_{x \to c^+} f(x)$ as the limit of $f(x)$ as x approaches c from the right.
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

E.g.:



as $x \to c$ from the right, $f(x) \to 2$

note: the $\lim_{x \to c} f(x)$ D.N.E., and $f(c) = 4$

but $\lim_{x \to c^+} f(x) = 2$.

One-sided limits

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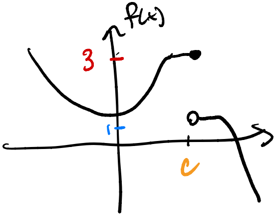
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

Similarly, $\lim_{x \to c^+} f(x)$ is the limit of $f(x)$ as x approaches c from the left.
This limit is L if $f(x)$ is close to L whenever x is close to and less than x.

Typo: should be $\lim_{x \to c^-} f(x)$

Eg:



$\lim_{x \to c^-} f(x) = 3$

$\lim_{x \to c^+} f(x) = 1$

$\lim_{x \to c} f(x)$ DNE.

One-sided limits

The last example motivates an opportunity to consider "one-sided" limits.

We'll define $\lim_{x \to c^+} f(x)$ as the limit of $f(x)$ as x approaches c from the right.
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

Similarly, $\lim_{x \to c^+} f(x)$ is the limit of $f(x)$ as x approaches c from the left.
This limit is L if $f(x)$ is close to L whenever x is close to and less than x.

One-sided limits need not exist, and even when they do, it may be the case that,

$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$

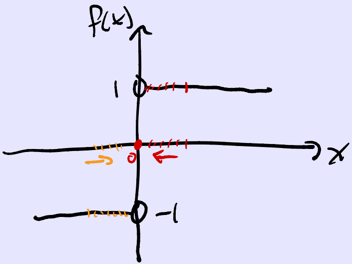
Example

Compute $\lim_{x \to 0^{\pm}} \frac{x}{|x|}$

(Ans: $+1, -1$, respectively)

From before, w/ $f(x) = \frac{x}{|x|}$,

we have



$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = -1$

$\Bigg\}$ not equal

Equality of one-sided limits

In the case that both one-sided limits are equal, a sensible conclusion holds.

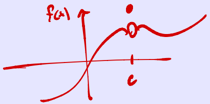
Theorem

The limit $\lim_{x \to c} f(x) = L$ *if and only if*

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

note: this does not mean $f(c) = L$

e.g.:



Why? Loosely, if f approaches the same value from the right and left, the f approaches that value in any shrinking window around $x = c$

Equality of one-sided limits

In the case that both one-sided limits are equal, a sensible conclusion holds.

Theorem

The limit $\lim_{x \to c} f(x) = L$ *if and only if*

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Why? When one-sided limits are equal to L, then $f(x)$ is close to L whenever x is close to c.

A summary

$$\lim_{x \to c} f(x) = L$$

– The numbers c and L should be finite real numbers. If L is not a real number, then the limit doesn't exist. i.e., if L is infinite

– In general the value $f(c)$ is *irrelevant* in determining the actual limit value L. (But it can be very suggestive!)

– Limits don't exist at vertical asymptotes or jump discontinuities.

– We can consider one-sided limits $\lim_{x \to c^+} f(x)$, or $\lim_{x \to c^-} f(x)$.

Additional exercises

You Try!

29. For the function f graphed in Figure 11, find the indicated limit or function value, or state that it does not exist.

(a) $\lim_{x \to -3} f(x)$ 2 (b) $f(-3)$ 1 (c) $f(-1)$ not defined

(d) $\lim_{x \to -1} f(x)$ 2.5 (e) $f(1)$ 2 (f) $\lim_{x \to 1} f(x)$ DNE

(g) $\lim_{x \to 1^-} f(x)$ 2 (h) $\lim_{x \to 1^+} f(x)$ 1 (i) $\lim_{x \to -1^+} f(x)$ 2.5

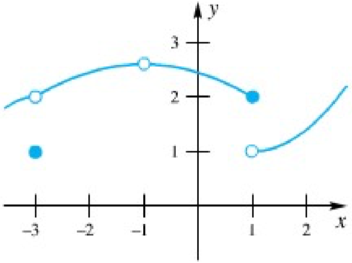
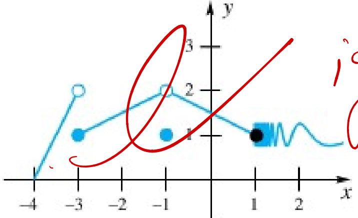


Figure 11



ignore.

Figure 12

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.