Math 1210: Calculus I
Introduction to limits

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.1

Limits

Calculus is the study of *limits*.

Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

Limits

Calculus is the study of *limits*.

Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

For many functions, the "answer" to this question is relatively boring:

Example

Let $f(x) = 2x + 3$. What should the "limit" of $f(x)$ be as x gets close to 1?
Ans: 5

Limits

Calculus is the study of *limits*.

Formally for us: what happens to values of a function $f(x)$ as x gets close to some value, $x \to c$?

The task of computing limits seems somewhat contrived, but is *extremely* useful in various practical scientific scenarios.

Why limits?

Here are a couple of examples about why limits are useful.

Example

Suppose a car's position along a straight road as a function of time is described by the function $f(t)$, where t is time.
Limits are useful for computing *velocity*.

Why limits?

Here are a couple of examples about why limits are useful.

Example

We know the circumference of a circle of radius r (it's $2\pi r$).
If we didn't know this formula, we could compute the circumference of a circle by inscribing a polygon, computing the exact perimeter of this polygon, and using limits.

Limit concepts and notation

We'll use the notation,

$$\lim_{x \to c} f(x),$$

to denote "the limit of f as x approaches c".

For now, c should be some (finite) real number.

Limit concepts and notation

We'll use the notation,

$$\lim_{x \to c} f(x),$$

to denote "the limit of f as x approaches c".

For now, c should be some (finite) real number.

If the value of this limit is some number L (another finite real number), that means that whenever x is close but not equal to c, then the value of $f(x)$ must be close to L.

The value $f(c)$ need not be equal to L, and $f(c)$ need not even be defined!

The limit $\lim_{x \to c} f(x)$ cannot have more than one value. Either it has a unique value, or it "does not exist".

A simple example

Example

Compute $\lim_{x \to 3} (x^3 - 3)$

(Ans: 24)

A nontrivial example

Example

Compute $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

(Ans: 2)

Example

Compute $\lim_{x \to 0} \frac{1}{x}$

(Ans: does not exist)

A harder example

Example

Compute $\lim_{x \to 0} \frac{\sin x}{x}$

(Ans: 1)

Infinitely many oscillations

Example

Compute $\lim_{x \to 0} \sin \frac{1}{x}$ (Ans: does not exist)

Jump discontinuities

Example

Compute $\lim_{x \to 0} \frac{x}{|x|}$ (Ans: does not exist)

One-sided limits

The last example motivates an opportunity to consider "one-sided" limits.

We'll define $\lim_{x \to c^+} f(x)$ as the limit of $f(x)$ as x approaches c from the right.
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

One-sided limits

The last example motivates an opportunity to consider "one-sided" limits.

We'll define $\lim_{x \to c^+} f(x)$ as the limit of $f(x)$ as x approaches c from the right.
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

Similarly, $\lim_{x \to c^+} f(x)$ is the limit of $f(x)$ as x approaches c from the left.
This limit is L if $f(x)$ is close to L whenever x is close to and less than x.

One-sided limits

The last example motivates an opportunity to consider "one-sided" limits.

We'll define $\lim_{x \to c^+} f(x)$ as the limit of $f(x)$ as x approaches c from the right.
I.e., this limit is L if $f(x)$ is close to L whenever x is close to and greater than x.

Similarly, $\lim_{x \to c^+} f(x)$ is the limit of $f(x)$ as x approaches c from the left.
This limit is L if $f(x)$ is close to L whenever x is close to and less than x.

One-sided limits need not exist, and even when they do, it may be the case that,

$$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$

Jump discontinuities, redux

Example

Compute $\lim_{x \to 0^{\pm}} \frac{x}{|x|}$

(Ans: $+1, -1$, respectively)

Equality of one-sided limits

In the case that both one-sided limits are equal, a sensible conclusion holds.

Theorem

The limit $\lim_{x \to c} f(x) = L$ *if and only if*

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Equality of one-sided limits

In the case that both one-sided limits are equal, a sensible conclusion holds.

Theorem

The limit $\lim_{x\to c} f(x) = L$ *if and only if*

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

Why? When one-sided limits are equal to L, then $f(x)$ is close to L whenever x is close to c.

A summary

$$\lim_{x \to c} f(x) = L$$

- The numbers c and L should be finite real numbers. If L is not a real number, then the limit doesn't exist.
- In general the value $f(c)$ is *irrelevant* in determining the actual limit value L. (But it can be very suggestive!)
- Limits don't exist at vertical asymptotes or jump discontinuities.
- We can consider one-sided limits $\lim_{x \to c^+} f(x)$, or $\lim_{x \to c^-} f(x)$.

29. For the function f graphed in Figure 11, find the indicated limit or function value, or state that it does not exist.

(a) $\lim_{x \to -3} f(x)$ (b) $f(-3)$ (c) $f(-1)$

(d) $\lim_{x \to -1} f(x)$ (e) $f(1)$ (f) $\lim_{x \to 1} f(x)$

(g) $\lim_{x \to 1^-} f(x)$ (h) $\lim_{x \to 1^+} f(x)$ (i) $\lim_{x \to -1^+} f(x)$

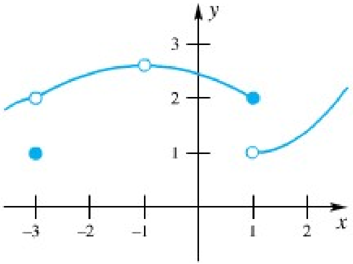

Figure 11

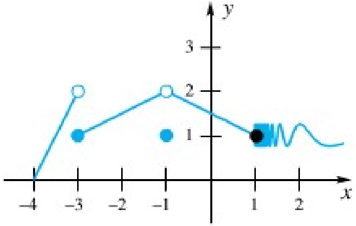

Figure 12

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.