Math 1210: Calculus I
Rigorous definition of limits

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.2

We have discussed the concept of a *limit*,

$$\lim_{x \to c} f(x) = L,$$

where L may not exist in some cases. E.g, we saw $\lim_{x \to 0} \frac{x}{|x|}$ D.N.E.

Such a limit exists when the value of $f(x)$ is very close to L whenever x is very close to c (but not equal to c). E.g., we saw $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$, but $f(x) = \frac{x^2-1}{x-1}$ is not defined at $x=1$

This concept of a limit as we've described it is not quite rigorous: it is conceptually understandable, but not logically precise. So far, we've used graphical arguments. These are extremely useful for the Calculus you'll be doing.

Today: We'll describe the precise underpinnings of limits.

↳ you won't be tested on this, but the rigorous def'n of the limit took 200 years to solidify and is worth seeing

A path to precision

We want

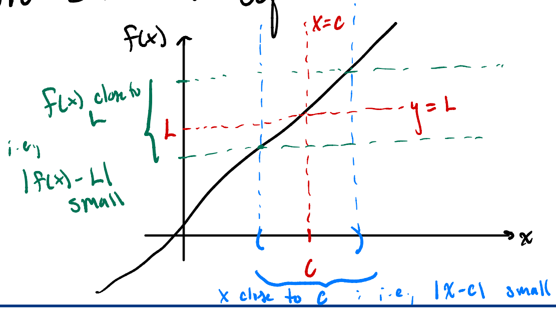
$$\lim_{x \to c} f(x) = L,$$

to mean that "when x is close to c, then $f(x)$ should be close to L".

It's more convenient to flip this statement around a bit, and instead ask that,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

The picture is the same as before:



A path to precision

We want

$$\lim_{x \to c} f(x) = L,$$

to mean that "when x is close to c, then $f(x)$ should be close to L".

It's more convenient to flip this statement around a bit, and instead ask that,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

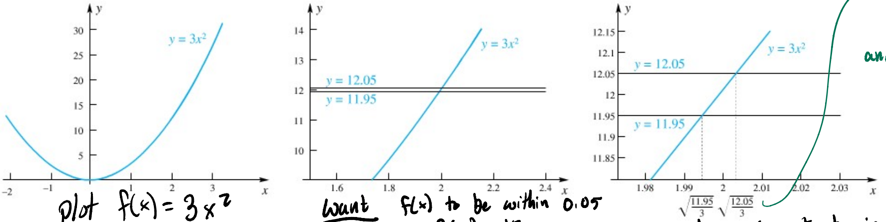
note: $f(x) = 3x^2 = y$

$\Rightarrow x = \sqrt{\frac{y}{3}}$

so if $y = 11.95$, $x = \sqrt{\frac{11.95}{3}}$

and similarly for $y = 12.05$

Consider the statement $\lim_{x \to 2} 3x^2 = 12$.



plot $f(x) = 3x^2$

want $f(x)$ to be within 0.05 of $f(2) = 12$

See that x must be in the interval pictured

Figure 1

Another view of the same example

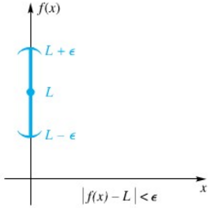
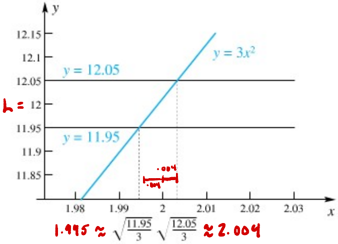


Figure 2



Let $L = 12$; $c = 2$.

<u>Given</u> $\varepsilon = 0.05$, we see that

if we want $|f(x) - L|$ to be less than ε,

choose x so that $|x-2| < 0.004$

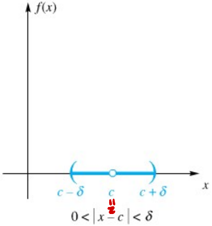


Figure 3

We want to be able to do this for <u>any</u> $\varepsilon > 0$

(so we try to force $f(x)$ to become <u>arbitrarily</u> close to L)

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

"$f(x)$ arbitrarily close to L" For any proximity parameter $\epsilon > 0$

no matter how small

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

"$f(x)$ arbitrarily close to L"	For any proximity parameter $\epsilon > 0$
"x close to c"	there is another proximity parameter $\delta > 0$

specified closeness

Some δ, which may depend on ε

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

"$f(x)$ arbitrarily close to L"	For any proximity parameter $\epsilon > 0$
"x close to c"	there is another proximity parameter $\delta > 0$
"by restricting x to be close to c"	whenever $\underline{0 <} \, |x - c| < \delta$

x close to c but not equal to c.

Remember: limits don't depend on the value of f at c, just 'near' c

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

"$f(x)$ arbitrarily close to L"	For any proximity parameter $\epsilon > 0$
"x close to c"	there is another proximity parameter $\delta > 0$
"by restricting x to be close to c"	whenever $0 < \lvert x - c \rvert < \delta$
"we can make $f(x)$ arbitrarily close to L"	then $\lvert f(x) - L \rvert < \epsilon$

↳ dont worry if $f(x) = L$.
That's fine!
Then we'd have

$$0 < \varepsilon$$

which is exactly what
we assume to be true
of ε.

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

"$f(x)$ arbitrarily close to L"	For any proximity parameter $\epsilon > 0$
"x close to c"	there is another proximity parameter $\delta > 0$
"by restricting x to be close to c"	whenever $0 < \|x - c\| < \delta$
"we can make $f(x)$ arbitrarily close to L"	then $\|f(x) - L\| < \epsilon$

This yields a definition. Putting it all together...

Definition

The statement $\lim_{x \to c} f(x) = L$ means that for any given $\epsilon > 0$, we can find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

NB: The lower inequality in $0 < |x - c| < \delta$ is important! Without it, we allow $x = c$, which is not the intent of a limit.

Usage of the definition

$$\lim_{x\to c} f(x) = L \text{ means that}$$

$$\text{for any given } \epsilon > 0\text{, we can find a } \delta > 0 \text{ so that if } 0 < |x-c| < \delta\text{, then } |f(x) - L| < \epsilon$$

Some observations:

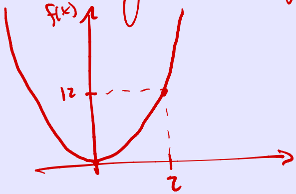
- This definition is like a game against an adversary: the adversary picks $\epsilon > 0$, and you must provide a $\delta > 0$ that works. We did this in the previous example with $\varepsilon = 0.05$. What if they'd chosen $\varepsilon = 0.005$? or $\varepsilon = 0.0000005$? By leaving $\varepsilon > 0$ arbitrary, we account for all of these proximity parameters at once.
- To (rigorously) show that a limit is true, one must
  1. assume an arbitrary $\epsilon > 0$ is given
  2. algebraically manipulate $f(x)$ to find a δ satisfying the desired inequalities
  3. δ will depend on the choice of ϵ.

Example (Definition of a limit)

Prove that $\lim_{x\to 2} 3x^2 = 12$.

(Answer is a paragraph narrative, with one choice $\delta(\epsilon) = \min\{1, \cancel{\epsilon/9}\ \epsilon/15\}$.)

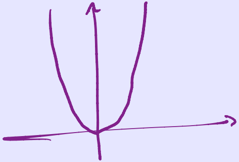
A simple graphical argument



is convincing, but it does not constitute a mathematical proof.

When graphical arguments can be misleading:

Consider $\lim_{x\to 2}\left(3x^2 - 0.00001\right)$



$\lim_{x\to 0}\left(3x^2 - 0.00001\right) = 0.0001 \neq 0$.

Goal: Prove $\lim_{x \to 2} 3x^2 = 12$.

Need the definition of the limit:

$\lim_{x \to c} f(x) = L \iff$ for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Above, $c = 2$, $L = 12$, and $f(x) = 3x^2$.

Preliminary work: want $|f(x) - L| = |3x^2 - 12| < \varepsilon$ for $|x - 2|$ small

Re-arrange:

$$|3x^2 - 12| < \varepsilon$$

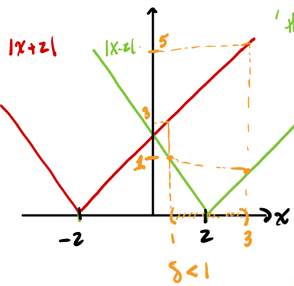
'the same as'

$$\iff 3|x^2 - 4| < \varepsilon$$

$$\iff |x^2 - 4| < \frac{\varepsilon}{3}$$

$$\iff \underbrace{|x+2|}_{\text{how to force this small?}}\ \underbrace{|x-2|}_{\text{controlled by our choice of } \delta} < \frac{\varepsilon}{3}$$

i.e., we will restrict to $|x-2| < \delta$



Limit $\delta < 1$

Then $|x - 2| \le 1$

$\Rightarrow 1 < x < 3$

So $|x + 2| < 5$

Graphically

So, choosing $\delta < 1$, we have

$$|3x^2 - 12| < \varepsilon$$

$$\iff |x+2|\,|x-2| < \frac{\varepsilon}{3}$$

$\delta \le 1$

$$\iff 5|x-2| < \frac{\varepsilon}{3}$$

$$\iff |x-2| < \frac{\varepsilon}{15}$$

Now, choose $\delta < \min\left\{1, \frac{\varepsilon}{15}\right\}$

With our choice, $\delta = \min\left\{1, \frac{\varepsilon}{15}\right\}$, we have an 'easy' proof:

Let $\varepsilon > 0$ be given. Choose $\delta < \min\left\{1, \frac{\varepsilon}{15}\right\}$. (i.e. $\delta = \frac{1}{2}\min\left\{1, \frac{\varepsilon}{15}\right\}$)

Then if $0 < |x - 2| < \delta$, we have

$$|f(x) - L| = |3x^2 - 12|$$

$$= 3|x^2 - 4|$$

$$= 3|x+2|\,|x-2|$$

since $\delta \le 1$ $\quad < 3(5)\,|x-2|$

$$= 15\,|x-2|$$

since $|x-2| < \delta < \frac{\varepsilon}{15}$ $\quad < 15 \cdot \frac{\varepsilon}{15}$

$$= \varepsilon$$

That is, $|x - 2| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$.

for any $\varepsilon > 0$.

The definition in practice

The previous exercise should convince you that, even for simple functions, proving

$$\lim_{x \to c} f(x) = L,$$

which amounts to showing that

For any given $\epsilon > 0$, we can find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$,

is a technical and possibly unpleasant task.

Additionally: such proofs require us to know the value of L beforehand!

There are more practically useful results that allow us to (rather easily) manipulate expressions to easily compute limits.

You won't need to return to these ε's & δ's
but do rember how they make limits rigorous.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.