Math 1210: Calculus I
Rigorous definition of limits

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.2

Limits

We have discussed the concept of a *limit*,

$$\lim_{x \to c} f(x) = L,$$

where L may not exist in some cases.

Such a limit exists when the value of $f(x)$ is very close to L whenever x is very close to c (but not equal to c).

This concept of a limit as we've described it is not quite rigorous: it is conceptually understandable, but not logically precise.

Today: We'll describe the precise underpinnings of limits.

A path to precision

We want

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to mean that "when x is close to c, then $f(x)$ should be close to L".

It's more convenient to flip this statement around a bit, and instead ask that,

We can make $f(x)$ arbitrarily close to L by restricting x to be close to c

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Consider the statement $\lim_{x \to 2} 3x^2 = 12$.

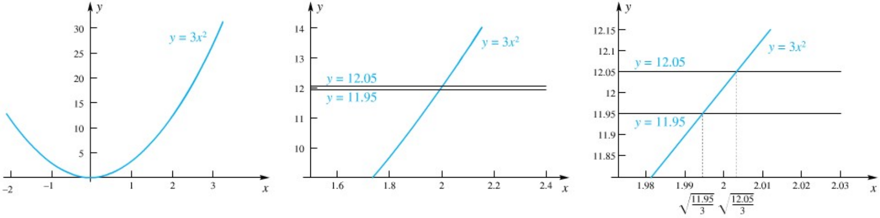


Figure 1

Another view of the same example

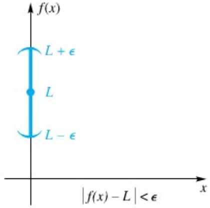
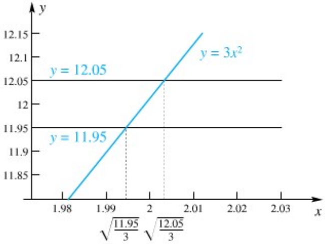


Figure 2



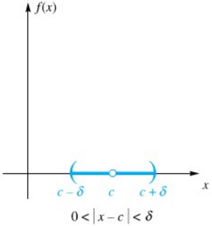


Figure 3

Toward a definition

Hence, we can codify our desired precise definition of the limit by quantifying the statement,

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"$f(x)$ arbitrarily close to L"	For any proximity parameter $\epsilon > 0$

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"we can make $f(x)$ arbitrarily close to L"	then $\lvert f(x) - L \rvert < \epsilon$

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This yields a definition.

Definition

The statement $\lim_{x \to c} f(x) = L$ means that for any given $\epsilon > 0$, we can find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

NB: The lower inequality in $0 < |x - c| < \delta$ is important! Without it, we allow $x = c$, which is not the intent of a limit.

Usage of the definition

$$\lim_{x \to c} f(x) = L \text{ means that}$$

for any given $\epsilon > 0$, we can find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$

Some observations:

- This definition is like a game against an adversary: the adversary picks $\epsilon > 0$, and you must provide a $\delta > 0$ that works.
- To (rigorously) show that a limit is true, one must
  1. assume an arbitrary $\epsilon > 0$ is given
  2. algebraically manipulate $f(x)$ to find a δ satisfying the desired inequalities
  3. δ will depend on the choice of ϵ.

An example

Example (Definition of a limit)

Prove that $\lim_{x \to 2} 3x^2 = 12$.

(Answer is a paragraph narrative, with one choice $\delta(\epsilon) = \min\{1, \epsilon/9\}$.)

The definition in practice

The previous exercise should convince you that, even for simple functions, proving

$$\lim_{x \to c} f(x) = L,$$

which amounts to showing that

For any given $\epsilon > 0$, we can find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$,

is a technical and possibly unpleasant task.

Additionally: such proofs require us to know the value of L beforehand!

There are more practically useful results that allow us to (rather easily) manipulate expressions to easily compute limits.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.