Math 1210: Calculus I
Limit theorems

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.3

Limits

We have discussed the concept of a *limit*,

$$\lim_{x \to c} f(x) = L,$$

where L may not exist in some cases.

Such a limit exists when the value of $f(x)$ is very close to L whenever x is very close to c (but not equal to c).

We have also introduced one-sided limits,

$$\lim_{x \to c^+} f(x) = r, \qquad \lim_{x \to c^-} f(x) = \ell,$$

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We've also seen that computing these limits is conceptually understandable, but can be cumbersome in practice.

Today we'll introduce some theorems involving limits that make computation of limits much more approachable.

Arithmetic operations on limits

One of our main results is the following theorem.

Theorem (Arithmetic operations on limits)

Let f and g be functions with limits at c. Let k be any constant, and let n be a positive integer. Then:

1. $\lim_{x \to c} k = k$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$
4. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
5. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
6. $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

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7. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, *assuming* $\lim_{x \to c} g(x) \neq 0$.

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8. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$,

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8. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$,
9. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, *assuming* $\lim_{x \to c} f(x) > 0$ *when* n *is even.*

Examples galore

Example

Compute $\lim_{x \to 2} 3x^2$.

(Ans: 12)

Examples galore

Example

Compute $\lim_{x \to 1} \left(4x - 5x^3\right)$

(Ans: -1)

Example

Compute $\lim_{x \to 3} \frac{\sqrt[3]{x^2 + 18}}{2x}$

(Ans: $\frac{1}{2}$)

Example

Compute $\lim_{x \to 3} \sqrt{f(x)}\, g^3(x)$, assuming $\lim_{x \to 3} f(x) = 4$ and $\lim_{x \to 3} g(x) = -2$.
(Ans: -16)

Substitution

All the previous examples are rather anticlimactic: you can verify that we reach the same answers by simply *plugging in* the value $x = c$ into $f(x)$.

This *substitution* strategy is an extraordinarily useful tool that we can exercise.

Theorem (Substitution for limits)

Suppose that f is a polynomial or rational function. Then

$$\lim_{x \to c} f(x) = f(c),$$

assuming f is defined at c.
(Polynomials are always well-defined for any c, and rational functions can only be undefined at c if the denominator vanishes at c.)

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Recall: a polynomial g has the form $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some constants $a_n, a_{n-1}, \ldots, a_1, a_0$.

A function f is rational if it has the form $f(x) = \frac{g(x)}{h(x)}$, for polynomials g and h, the numerator and denominator, respectively.

Polynomial/rational limit examples

Example

Compute

$$\lim_{x \to -3} \frac{x^3 + 15}{2x^2 - 5x + 3}$$

(Ans: $-\frac{1}{3}$)

Polynomial/rational limit examples

Example

Compute

$$\lim_{x \to 2} \frac{x^2 + 1}{x - 2}$$

(Ans: does not exist)

Canceling common factors

We've seen that sometimes factoring helps to make sense of limits of rational functions when substitution fails:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1}$$

That we can cancel common factors to remove problematic $\frac{0}{0}$ expressions is given below.

Theorem (Canceling common factors)

Suppose $f(x) = g(x)$ for all x contained in an interval around the point c, but possibly $f(c) \neq g(c)$. Then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

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Using this theorem, we conclude that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \stackrel{(*)}{=} \lim_{x\to 1} x+1 = 2.$$

where $(*)$ exercises this theorem by noting that $x+1 = \frac{(x-1)(x+1)}{x-1}$ except when $x = 1$.

The squeeze theorem

Here's a final result for limits.

Suppose that we wish to compute $\lim_{x \to c} g(x)$, and that,

$$f(x) \leqslant g(x) \leqslant h(x)$$

for all x close to but not equal to c.

If we happen to know that $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then it seems plausible that g should have this same limit.

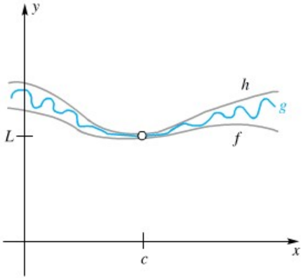

Figure 2

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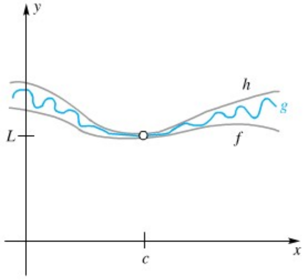

Figure 2

Theorem (Squeeze Theorem)

Suppose f, g, and h are functions satisfying $f(x) \leqslant g(x) \leqslant h(x)$ for all x close to but not equal to c. If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. MyMathLab Series. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.