Announcements:

- Labs
  - Two lowest lab grades dropped
  - Grading for Labs takes ~1 week
- Office Hours
  - Mon 9:30-10:30, WEB 4666
  - Thu 1:30-2:30, LCB 116
- Hw #1 due Friday
  - "Practice" uploading to gradescope if it's a new system for you!

Math 1210: Calculus I
Trigonometry Review

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 0.7

Trigonometry

Trigonometry is the mathematical study of angles and their relations to lengths of triangles.

Trigonometry is a basic tool for analyzing simple geometry, and we'll make substantial use of it in this course.

We start with the fundamental definition of the "sine" and "cosine" functions.

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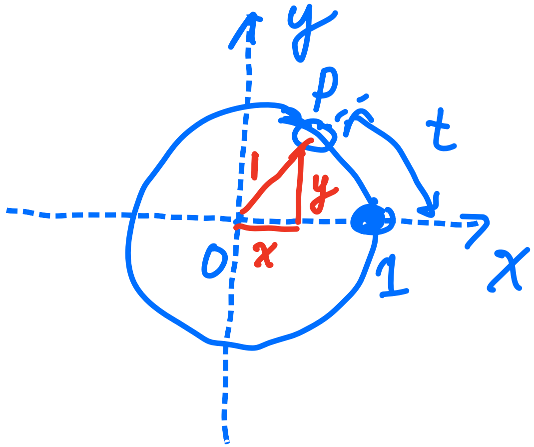

- We consider the unit circle in the plane.
- We know the circumference of this circle is 2π. ("$2\pi r$")
- Let t satisfy $0 \leqslant t \leqslant 2\pi$.
- We consider a point P on the unit circle that is a distance of t counterclockwise along the circle starting from the point $(1, 0)$.
- We'll also consider the (right) triangle whose vertices are the origin, P, and the projection of P onto the horizontal (x) axis.

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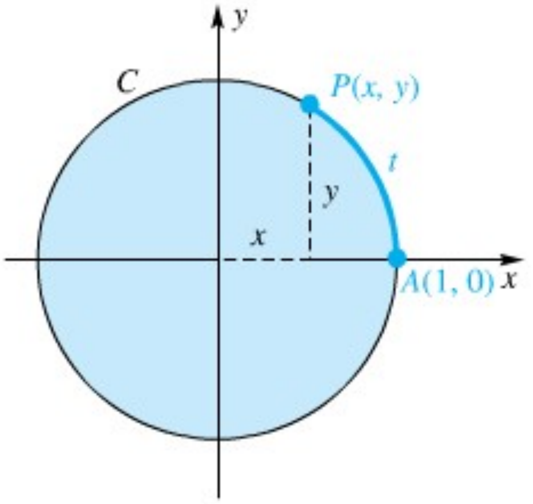
- We'll also consider the (right) triangle whose vertices are the origin, P, and the projection of P onto the horizontal (x) axis.
- For a fixed t, we let (x, y) be the coordinates of P.



The unit circle

Figure 2

The sine and cosine functions

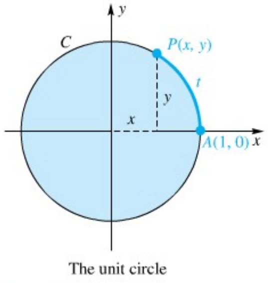


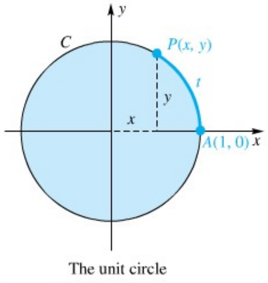
Figure 2

Definition

Under the previous setup, $\cos t = x$, and $\sin t = y$.

The definition applies to any real number t, even negative t (clockwise movement along circle).

The sine and cosine functions



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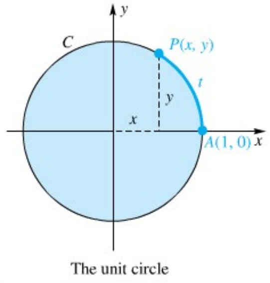
- These functions are *periodic*: $\sin t = \sin(t + 2\pi)$ and $\cos t = \cos(t + 2\pi)$ for any t.
  (This is because the unit circle has circumference 2π so $P(t) = P(t + 2\pi)$).

The sine and cosine functions



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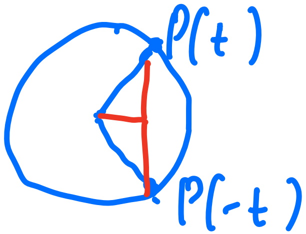
- When comparing $P(t)$ versus $P(-t)$, only y changes by a negative sign. Hence: $\cos(-t) = \cos(t)$ and $\sin(-t) = -\sin(t)$, i.e., $\cos$ is an even function, and $\sin$ is an odd function.



The sine and cosine functions

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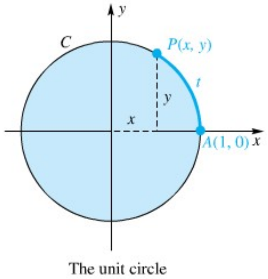

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- Because $\sin t$ and $\cos t$ are the lengths of the two smaller sides of a hypotenuese-1 triangle,

$$\sin^2 t + \cos^2 t = 1 \qquad \text{for any value of } t$$

$$(\sin t)^2 + (\cos t)^2 = 1$$

Some values for sine, cosine

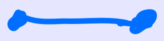
Example

Using the geometric interpretation for sine, cosine, compute values for,

$$\sin 0 \qquad \sin\frac{\pi}{4} \qquad \sin\frac{\pi}{2} \qquad \sin\frac{3\pi}{4} \qquad \sin\pi$$

$$\cos 0 \qquad \cos\frac{\pi}{4} \qquad \cos\frac{\pi}{2} \qquad \cos\frac{3\pi}{4} \qquad \cos\pi.$$

$t = 0$

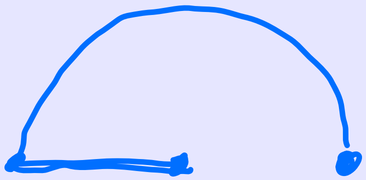


$(y=0)$

0 $(1,0) = P = (x,y)$

$\cos 0 = 1, \quad \sin 0 = 0$

$t = \pi$

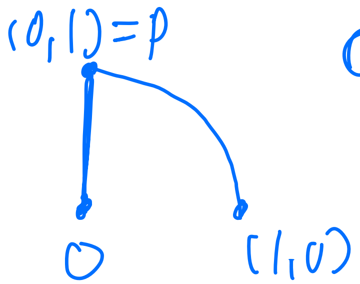


$P(\pi)$ 0 $(1,0)$

$\| $

$(-1,0) = (x,y), \quad \cos\pi = -1, \quad \sin\pi = 0$

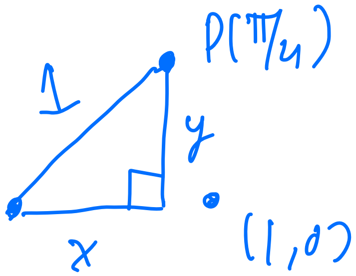
$t = \pi/2$ (A quarter around circle)



$\cos \pi/2 = 0$

$\sin \pi/2 = 1$

$t = \pi/4$



$x = y$ (e.g., because non-90° angles are equal).

$$x^2 + y^2 = 1$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\sqrt{\frac{1}{2}}$$

From figure: $x > 0$

$$\Rightarrow x = +\sqrt{\tfrac{1}{2}} = y$$

$$\cos \pi/4 = \sin \pi/4 = \sqrt{\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{2}$$

$\sin(3\pi/4)$, $\cos(3\pi/4)$

as exercise.

More values for sine, cosine

t	$\sin t$	$\cos t$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$
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The values of sine, cosine in the table are worth memorizing.

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These values allow us to graph these functions:

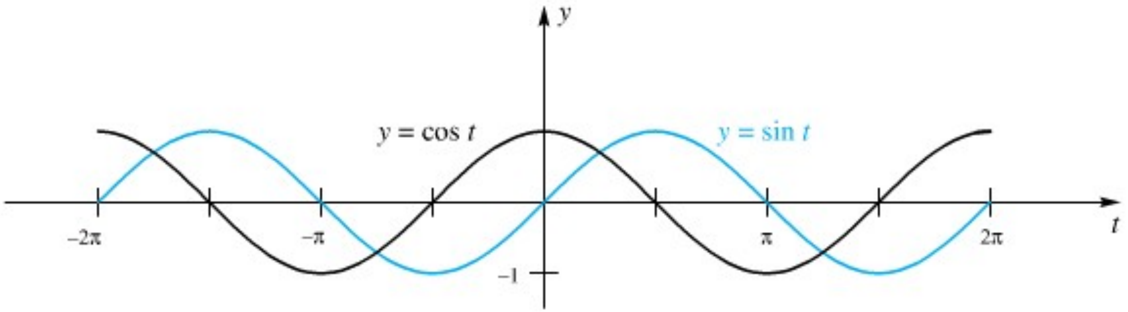
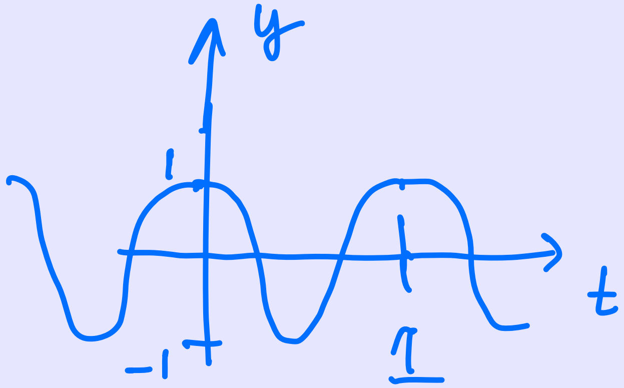


Figure 6

Example

Sketch the graph of $y = \cos(2\pi t)$.

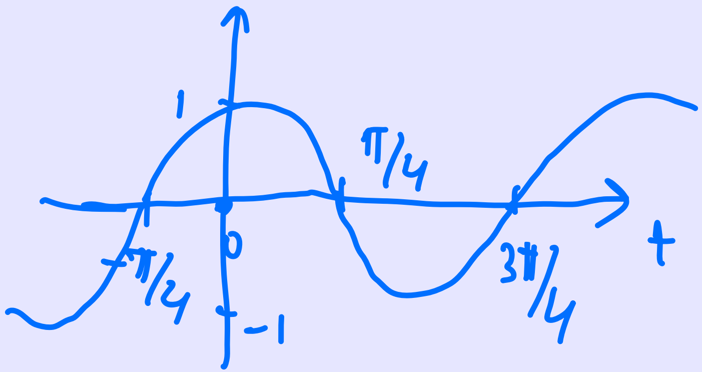


Example

Sketch the graph of $y = \sin\left(2t + \frac{\pi}{2}\right)$.

$2t + \pi/2$ takes values in $[0, 2\pi]$ when

t lies in $[-\pi/4, 3\pi/4]$



"Derived" trig functions

We can use $\sin$ and $\cos$ to define other trig functions:

$$\tan t = \frac{\sin t}{\cos t}, \qquad \cot t = \frac{\cos t}{\sin t}, \qquad \sec t = \frac{1}{\cos t}, \qquad \csc t = \frac{1}{\sin t}.$$

These four functions are not defined when their denominators vanish.

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Plotting these functions is a little more involved, but can still be done.

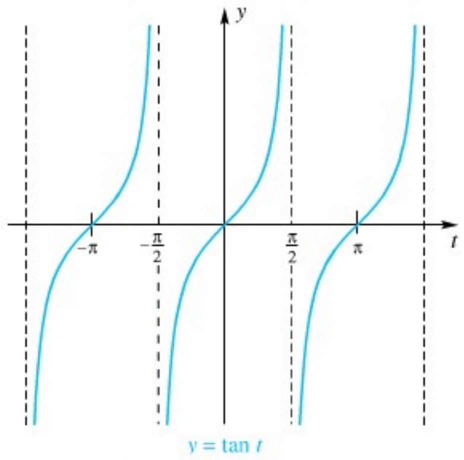


Figure 11

Example

Show that $1 + \tan^2 t = \sec^2 t$ for all t where the involved functions are defined.

$$\sin^2 t + \cos^2 t = 1$$

divide by $\cos^2 t$

$$\frac{\sin^2 t}{\cos^2 t} + 1 = \frac{1}{\cos^2 t} \longrightarrow \tan^2 t + 1 = \sec^2 t$$

An advanced example: putting some things together

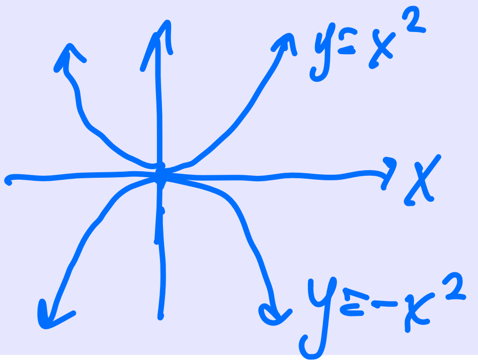
Example

Compute $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Recall: $\sin(1/x)$ has infinitely many oscillations near $x=0$.

Note: for any t: $-1 \leq \sin t \leq 1$ (cf. graphical def'n)

$$\Longrightarrow \underbrace{-x^2}_{\to 0 \text{ as } x \to 0} \leq x^2 \sin(1/x) \leq \underbrace{+x^2}_{\to 0 \text{ as } x \to 0}$$



Then by the squeeze theorem: $\lim_{x \to 0} x^2 \sin(1/x) = 0$

"Last time": $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

Recall some identities

Odd–even identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

Cofunction identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Addition identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Double-angle identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Half-angle identities

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.