Math 1210: Calculus I
Limits for trigonometric functions

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.4

Trigonometric functions

With a firmer understanding of trigonometric functions, we can delve into some limits involving these functions.

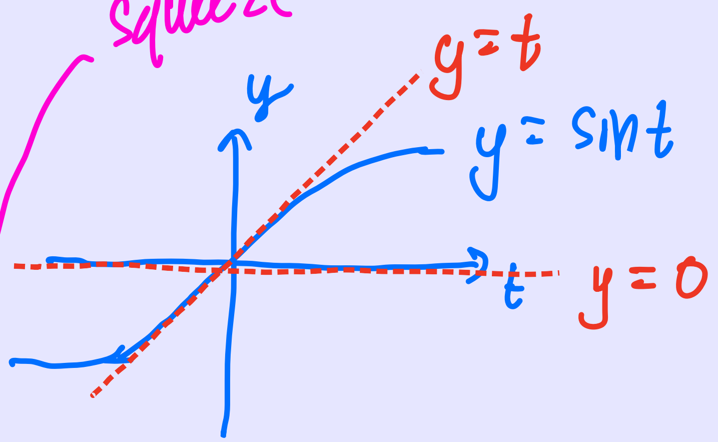
Example

Show that $\lim_{t\to 0} \sin t = 0$, and $\lim_{t\to 0} \cos t = 1$.
(Squeeze between 0 and t.)

Let's look at the first limit:

$y=0$ and $y=t$ bound ("squeeze") $y=\sin t$ near $t=0$.

squeeze theorem



Since $\lim_{t\to 0} 0 = 0$, $\lim_{t\to 0} t = 0 \implies \lim_{t\to 0} \sin t = 0$

Example

Show that $\lim_{t\to c} \sin t = \sin c$. (c an arbitrary, fixed constant)
(Substitute $t = c + h$, use angle sum formula)

$$\underbrace{\lim_{t\to c}}_{t \text{ is "around" } c} \sin t \overset{t=c+h}{=} \lim_{h\to 0} \sin(c+h) \overset{\text{angle sum identity}}{=} \lim_{h\to 0}\left[(\sin c)(\cos h) + (\sin h)(\cos c)\right]$$

$$= \left(\lim_{h\to 0} \sin c\right)\left(\lim_{h\to 0} \cos h\right) + \left(\lim_{h\to 0} \sin h\right)\left(\lim_{h\to 0} \cos c\right)$$

$$= (\sin c)\cdot 1 + 0 \cdot \cos c = \underline{\sin c} \checkmark$$

A more general result

We have seen that limits of the sine function can be evaluated through substitution. This is a generic property of trigonometric functions.

Theorem (Substitution for trigonometric limits)

As long as c *is in the domain of the functions below, then,*

$$\lim_{t\to c} \sin t = \sin c \qquad \lim_{t\to c} \cos t = \cos c \qquad \lim_{t\to c} \tan t = \tan c$$

$$\lim_{t\to c} \csc t = \csc c \qquad \lim_{t\to c} \sec t = \sec c \qquad \lim_{t\to c} \cot t = \cot c$$

Some special limits

With rational functions, if we encountered $\frac{0}{0}$ when performing substitution for a limit, we attempted to rectify this by factoring and canceling terms.

We cannot do something quite so simple with trig functions. However, there are some special limits to note that will be very useful.

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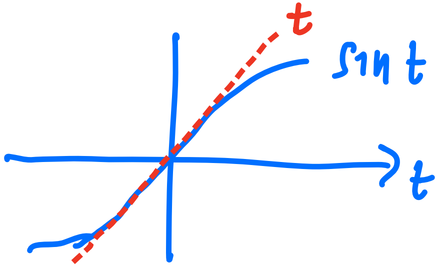
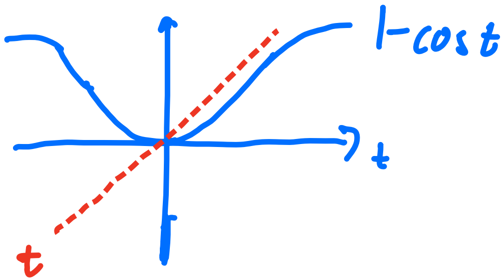
Theorem (Special trigonometric limits)

The following limits hold:

$$\lim_{t \to 0} \frac{\sin t}{t} = 1, \qquad \lim_{t \to 0} \frac{1 - \cos t}{t} = 0.$$

why?





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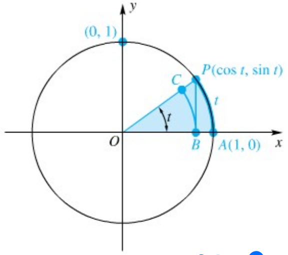


Figure 2

3 areas:
1.) Sector OBC
2.) Triangle OBP
3.) Sector OAP

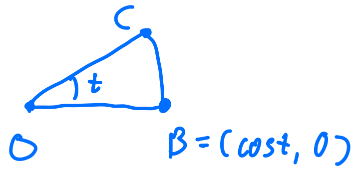
Note: area of a circular sector of radius r, sweeping out angle θ is $\frac{1}{2}\theta r^2$

Let's show the first limit:

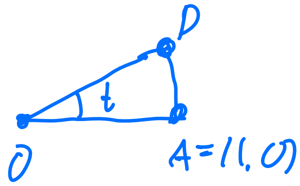
area(OBC) $\leq$ area(OBP) $\leq$ area(OAP)

Why?: area of circle is $\pi r^2 = \frac{1}{2}\underbrace{(2\pi)}_{\text{angle swept out}} r^2$

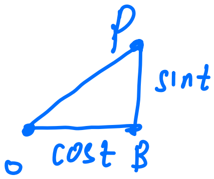
area(OBC): angle t
radius "$x = \cos t$"
area $= \frac{1}{2} t \cos^2 t$



area(OAP): angle t
radius 1
area: $\frac{1}{2} t\, 1^2$



area(OBP): width: $\cos t$
height: $\sin t$
area: $\frac{1}{2} \sin t \cos t$



$$\text{area}(OBC) \le \text{area}(OBP) \le \text{area}(OAP)$$

$$\frac{1}{2} t \cos^2 t \le \frac{1}{2} \sin t \cos t \le \frac{1}{2} t$$

$\downarrow$ multiply by 2, divide by $t \cos t$

$$\cos t \le \frac{\sin t}{t} \le \frac{1}{\cos t}$$

$\downarrow$ 1 as $t \to 0$

$\downarrow$ 1 as $t \to 0$

$$\Rightarrow \lim_{t \to 0} \frac{\sin t}{t} = 1$$

Ex. Show, using the previous limit, that

$$\lim_{t\to 0} \frac{1-\cos t}{t} = 0$$

$$\lim_{t\to 0} \frac{(1-\cos t)}{t} \frac{(1+\cos t)}{(1+\cos t)} = \lim_{t\to 0} \frac{1-\cos^2 t}{t(1+\cos t)}$$

$$= \lim_{t\to 0} \frac{\sin^2 t}{t(1+\cos t)}$$

$$= \lim_{t\to 0} \left(\frac{\sin t}{t}\right)\left(\frac{\sin t}{(1+\cos t)}\right)$$

$$= \left(\lim_{t\to 0} \frac{\sin t}{t}\right)\left(\lim_{t\to 0} \frac{\sin t}{1+\cos t}\right)$$

$$= 1 \cdot \frac{\lim_{t\to 0} \sin t}{\left(\lim_{t\to 0} 1\right) + \left(\lim_{t\to 0} \cos t\right)}$$

$$= 1 \cdot \frac{0}{1+1} = 0$$

Evaluating limits

For limits involving polynomials, rational functions, root functions, or trigonometric functions, here is an ordered set of steps for evaluation:

1. Direct substitution: if the result is a valid number, that is the limit.
2. Rational cancellation: Attempt to factor + cancel terms to remove behavior like $\frac{0}{0}$
3. Use special trigonometric limits for $\frac{0}{0}$-like behavior stemming from trig functions

Example

Evaluate $\lim_{x\to 0} \dfrac{\sin 2x}{x}$.

(Ans: 2)

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{2x}\cdot 2 = 2\lim_{x\to 0} \frac{\sin 2x}{2x}$$

substitution: $t = 2x$

$x \to 0 \iff t \to 0$

$$2\lim_{x\to 0} \frac{\sin 2x}{2x} \overset{t=2x}{=} 2\lim_{t\to 0} \frac{\sin t}{t} = 2$$

Example

Evaluate $\lim_{\cancel{x\to 0}\; t\to 0} \dfrac{1-\cos t}{\sin t}$.

(Ans: 0)

$$\frac{1+\cos t}{1+\cos t} \rightarrow \frac{\sin t}{1+\cos t}$$

One strategy: $\lim_{t\to 0} \dfrac{1-\cos t}{\sin t} = \lim_{t\to 0} \dfrac{1-\cos t}{t}\cdot\dfrac{t}{\sin t}$

$$= \left(\lim_{t\to 0} \frac{1-\cos t}{t}\right)\left(\lim_{t\to 0}\frac{t}{\sin t}\right) = \left(\lim_{t\to 0} \frac{1-\cos t}{t}\right)\left(\lim_{t\to 0}\frac{1}{\frac{\sin t}{t}}\right)$$

$$= 0\cdot\frac{1}{1} = 0.$$

Example

Evaluate $\lim_{x \to 0} \dfrac{\tan x}{\sin 3x}$.

(Ans: $\frac{1}{3}$)

$$\lim_{x\to 0} \frac{\tan x}{\sin 3x} = \lim_{x\to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{\sin 3x} = \left(\lim_{x\to 0} \frac{1}{\cos x}\right)\left(\lim_{x\to 0} \frac{\sin x}{\sin 3x}\right)$$

$$= 1 \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{3\cancel{x}}{\sin 3x} \cdot \frac{1}{3}$$

$$= \frac{1}{3}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\frac{\sin 3x}{3x}}\right) = \frac{1}{3} \cdot 1 \cdot \frac{1}{1} = \frac{1}{3}.$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. MyMathLab Series. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.