Math 1210: Calculus I
Limits at infinity and infinite limits

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.5

A discussion on infinity, ∞

The meaning of "∞" depends on the context.

For this course, we mainly use ∞ to mean a conceptual quantity that is larger than any real number.

Conceptual is the operative word: ∞ is <u>not</u> a number!

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As a concept, a quantity like "$-\infty$" makes sense: it is a conceptual quantity smaller than any real number.

Things like "$3 \cdot \infty$" are conceptually equivalent to "∞".
(But avoid writing things like $3\infty = \infty$.)

Things like $\infty \cdot (-\infty)$ conceptually mean a very very large positive number multiplied by a very very small negative number. This *conceptually* should be a very very smal negative number, i.e., $-\infty$.
(Again, avoid writing "$\infty \cdot (-\infty) = -\infty$".)

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There are many quantities involving ∞ that do not make sense, and you should not attempt to interpret them without experience:

- $\infty - \infty$
- $\frac{\infty}{\infty}$
- $\frac{\infty}{0}$
- $0 \cdot \infty$

large: $x^2 - x$ vs. $(x+1) - x$ vs. $x - x$

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Overall, ∞ is a concept and you can + should treat it as such: any place you see an "∞", think "what would happen if I put an extremely large positive number here?"

If the answer is ambiguous, then do not attempt to go further and instead try another approach.

Part I: Limits "at" infinity

When discussing limits, $\lim_{x \to c} f(x)$, we have always assumed c is a (finite) real number.

Limits "at infinity" generalize this by using $c = +\infty$ or $c = -\infty$.

Conceptually, for $c = +\infty$, this just means that instead of asking what happens when x approaches c, we instead ask what happens when x becomes an extremely large positive number.

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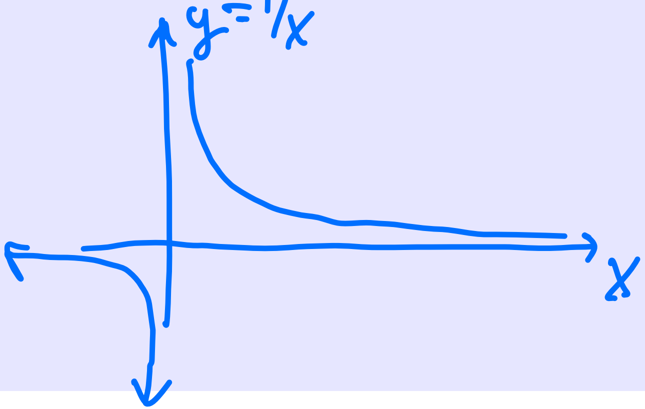
Example

Provide an argument for what value $\lim_{x \to \infty} \frac{1}{x}$ should have.



This limit "should" be 0.

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

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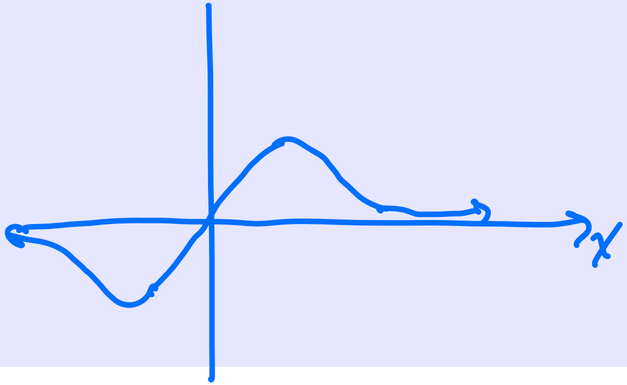
Provide an argument for what value $\displaystyle\lim_{x \to \infty} \frac{x}{1+x^2}$ should have.



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A formal definition

Like before, a strict logical definition for limits are infinity can be cumbersome, but their utility lies in understanding precisely what is meant by such limits.

Definition

Suppose $f(x)$ is defined for x sufficiently large (i.e., for all x larger than some real number x_0). Then $\lim_{x \to \infty} f(x) = L$ means: for any $\epsilon > 0$, there is some large number M such that if $x > M$, then $|f(x) - L| < \epsilon$.

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Example

Provide a definition for the statement $\lim_{x\to-\infty} f(x) = L$.

For any $\varepsilon > 0$, there is some large positive M such that if $x < -M$, then $|f(x) - L| < \varepsilon$.

A fundamental limit at ∞

Example

Let k be a positive integer. Show that $\lim_{x\to\infty} \frac{1}{x^k} = 0.$

(NB: by a similar argument, $\lim_{x\to-\infty} \frac{1}{x^k} = 0$ is true.)

Using the definition: want $\left|\frac{1}{x^k} - 0\right| < \varepsilon$

$$\frac{1}{|x|^k} < \varepsilon \Longrightarrow |x| > 1/\sqrt[k]{\varepsilon}$$

Let $\varepsilon > 0$ be given. Choose $M = 1/\sqrt[k]{\varepsilon}$. Then if $x > M$, then $\left|\frac{1}{x^k} - 0\right| < \varepsilon$

Example

Compute $\lim_{x\to\infty} \dfrac{x}{x^2+1}$.

(Ans: 0)

Direct substitution: $\frac{\infty}{\infty}$ ☹

$$\lim_{x\to\infty} \frac{x}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x}}{\left(\lim_{x\to\infty} 1\right) + \left(\lim_{x\to\infty} \frac{1}{x^2}\right)} = \frac{0}{1+0} = 0.$$

Example

Compute $\lim_{x\to-\infty} \frac{x^3+3x+4}{1-3x^3}$.

(Ans: $-\frac{1}{3}$)

$$\lim_{x\to \cancel{\infty}\,-\infty} \frac{x^3+3x+4}{1-3x^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\cancel{\infty}\,-\infty} \frac{1+\frac{3}{x^2}+\frac{4}{x^3}}{\frac{1}{x^3}-3}$$

$$= \frac{1+0+0}{0-3} = -1/3$$

Ex: $\lim\limits_{x \to \infty} \dfrac{5x^3+3}{14x^4-17}$

divide by x^4

$= \lim\limits_{x \to \infty} \dfrac{\frac{5}{x} + \frac{3}{x^4}}{14 - \frac{17}{x^4}} = \dfrac{0+0}{14-0} = 0$

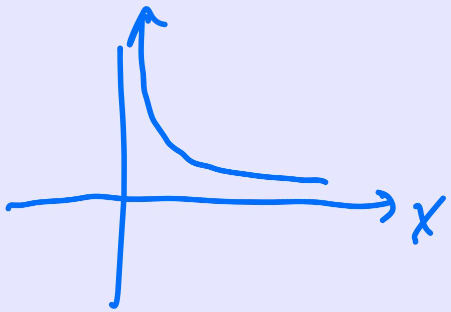
Part II: "Infinite" limits

To motivate infinite limits, consider the following example:

Example

Provide an argument for why $\lim_{x \to 0^+} \frac{1}{x} = \infty$ is a conceptually sensible statement.

(Also for $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.)



$\frac{1}{x}$ is arbitrarily large when x is positive and close to 0.

Infinite limits

Again, we cannot use our standard definition of limits to understand infinite limits. (Values cannot "approach ∞" in the traditional sense.)

Limit does not exist!

Definition

For a fixed real number c, the statement $\lim_{x \to c} f(x) = \infty$ means: For every $M > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$.

Similarly, $\lim_{x \to c^+} f(x) = \infty$ means: For every $M > 0$, there is a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) > M$.

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NB: Previously, if $f(x)$ took on unbounded values, we said that the limit did not exist. We are <u>still</u> saying that, but we are being more precise about *how* the limit fails to exist.

I.e., $\lim_{x \to c} f(x) = \infty$ means that the limit does not exist, and does not exist in a particular way.

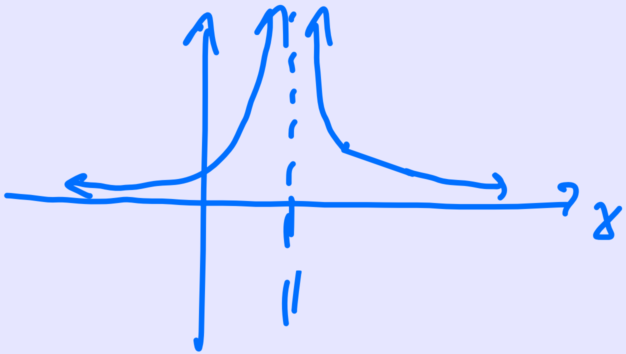
$f(x) = 1/x$ @ $x = 0$

Example

Compute $\lim_{x\to 1^-} \frac{1}{(x-1)^2}$ and $\lim_{x\to 1^+} \frac{1}{(x-1)^2}$.

(Ans: Both limits are ∞.) (Limits don't exist)

Geometrically:



$$\lim_{x\to 1^-} \frac{1}{(x-1)^2} = +\infty$$

$$\lim_{x\to 1^+} \frac{1}{(x-1)^2} = +\infty$$

Algebraically: $(x-1)^2 > 0$ away from $x=1$.

$\Rightarrow \frac{1}{(x-1)^2} \sim \frac{1}{\cancel{0}}$ near $x=1$ for small $\varepsilon \Rightarrow$ limits $\to \infty$

ε

Example

Compute $\lim_{x\to 2^+} \dfrac{x+1}{x^2-5x+6}$.
(Ans: $-\infty$)

Direct substitution fails (denominator is 0)

$$\lim_{x\to 2^+} \frac{x+1}{(x-2)(x-3)} = \underbrace{\left(\lim_{x\to 2^+} \frac{x+1}{x-3}\right)}_{-3} \underbrace{\left(\lim_{x\to 2^+} \frac{1}{x-2}\right)}_{\text{"}+\infty\text{"}}$$

$$= -\infty \text{ (doesn't exist)}$$

Asymptotes

Our notion of limits at infinity and infinite limits allows us to make precise definitions for concepts you've seen before: horizontal and vertical asymptotes to graphs.

Definition (Horizontal asymptote)

We say that the graph of $y = f(x)$ as a horizontal asymptote at $y = b$ if

$$\lim_{x \to \infty} f(x) = b \qquad \text{OR} \qquad \lim_{x \to -\infty} f(x) = b.$$

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Definition (Vertical asymptote)

We say that the graph of $y = f(x)$ as a vertical asymptote at $x = c$ if

$$\left|\lim_{x\to c^+} f(x)\right| = \infty \qquad \text{OR} \qquad \left|\lim_{x\to c^-} f(x)\right| = \infty.$$

Example

Identify any vertical and horizontal asymptotes for $y = \frac{3x}{x-1}$.
(Ans: Horizontal asymptote at $y = 3$, vertical asymptote at $x = 1$.)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. MyMathLab Series. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.