Math 1210: Calculus I
Limits at infinity and infinite limits

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.5

A discussion on infinity, ∞

The meaning of "∞" depends on the context.

For this course, we mainly use ∞ to mean a conceptual quantity that is larger than any real number.

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As a concept, a quantity like "$-\infty$" makes sense: it is a conceptual quantity smaller than any real number.

Things like "$3 \cdot \infty$" are conceptually equivalent to "∞".
(But avoid writing things like $3\infty = \infty$.)

Things like $\infty \cdot (-\infty)$ conceptually mean a very very large positive number multiplied by a very very small negative number. This *conceptually* should be a very very smal negative number, i.e., $-\infty$.
(Again, avoid writing "$\infty \cdot (-\infty) = -\infty$".)

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There are many quantities involving ∞ that do not make sense, and you should not attempt to interpret them without experience:

- $\infty - \infty$
- $\frac{\infty}{\infty}$
- $\frac{\infty}{0}$
- $0 \cdot \infty$

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Overall, ∞ is a concept and you can + should treat it as such: any place you see an "∞", think "what would happen if I put an extremely large positive number here?"

If the answer is ambiguous, then do not attempt to go further and instead try another approach.

Part I: Limits "at" infinity

When discussing limits, $\lim_{x \to c} f(x)$, we have always assumed c is a (finite) real number.

Limits "at infinity" generalize this by using $c = +\infty$ or $c = -\infty$.

Conceptually, for $c = +\infty$, this just means that instead of asking what happens when x approaches c, we instead ask what happens when x becomes an extremely large positive number.

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Example

Provide an argument for what value $\lim_{x \to \infty} \frac{x}{1 + x^2}$ should have.

A formal definition

Like before, a strict logical definition for limits are infinity can be cumbersome, but their utility lies in understanding precisely what is meant by such limits.

Definition

Suppose $f(x)$ is defined for x sufficiently large (i.e., for all x larger than some real number x_0). Then $\lim_{x \to \infty} f(x) = L$ means: for any $\epsilon > 0$, there is some large number M such that if $x > M$, then $|f(x) - L| < \epsilon$.

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Example

Provide a definition for the statement $\lim_{x \to -\infty} f(x) = L$.

A fundamental limit at ∞

Example

Let k be a positive integer. Show that $\lim_{x \to \infty} \frac{1}{x^k} = 0$.

(NB: by a similar argument, $\lim_{x \to -\infty} \frac{1}{x^k} = 0$ is true.)

Examples

Example

Compute $\lim_{x\to\infty} \frac{x}{x^2+1}$.

(Ans: 0)

Example

Compute $\lim_{x \to -\infty} \dfrac{x^3 + 3x + 4}{1 - 3x^3}$.

(Ans: $-\frac{1}{3}$)

Part II: "Infinite" limits

To motivate infinite limits, consider the following example:

Example

Provide an argument for why $\lim_{x \to 0^+} \frac{1}{x} = \infty$ is a conceptually sensible statement.

(Also for $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.)

Infinite limits

Again, we cannot use our standard definition of limits to understand infinite limits. (Values cannot "approach ∞" in the traditional sense.)

Definition

For a fixed real number c, the statement $\lim_{x \to c} f(x) = \infty$ means: For every $M > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$.

Similarly, $\lim_{x \to c^+} f(x) = \infty$ means: For every $M > 0$, there is a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) > M$.

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There are also similar definitions for limits equaling $-\infty$, and/or for $x \to c^-$.

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NB: Previously, if $f(x)$ took on unbounded values, we said that the limit did not exist. We are still saying that, but we are being more precise about *how* the limit fails to exist.

I.e., $\lim_{x \to c} f(x) = \infty$ means that the limit does not exist, and does not exist in a particular way.

Example

Compute $\lim_{x \to 1^-} \frac{1}{(x-1)^2}$ and $\lim_{x \to 1^+} \frac{1}{(x-1)^2}$.
(Ans: Both limits are ∞.)

Example

Compute $\lim_{x \to 2^+} \dfrac{x+1}{x^2 - 5x + 6}$.
(Ans: $-\infty$)

Our notion of limits at infinity and infinite limits allows us to make precise definitions for concepts you've seen before: horizontal and vertical asymptotes to graphs.

Definition (Horizontal asymptote)

We say that the graph of $y = f(x)$ as a horizontal asymptote at $y = b$ if

$$\lim_{x \to \infty} f(x) = b \quad \text{OR} \quad \lim_{x \to -\infty} f(x) = b.$$

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Definition (Vertical asymptote)

We say that the graph of $y = f(x)$ as a vertical asymptote at $x = c$ if

$$\left| \lim_{x \to c^+} f(x) \right| = \infty \qquad \text{OR} \qquad \left| \lim_{x \to c^-} f(x) \right| = \infty.$$

Example

Identify any vertical and horizontal asymptotes for $y = \frac{3x}{x-1}$.
(Ans: Horizontal asymptote at $y = 3$, vertical asymptote at $x = 1$.)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. MyMathLab Series. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.