

# Math 1210: Calculus I

## Continuity of functions

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.6

We've discussed motivation, definitions, and computations for limits, in particular for the statements,

$$\lim_{x \rightarrow c} f(x) = L,$$

$$\lim_{x \rightarrow c^-} f(x) = L_-,$$

where  $c$  can be a real number or  $\pm\infty$ .

If  $L$  or  $L_-$  are real numbers, then the limit exists.

The values  $L$  or  $L_-$  can be  $+\infty$  or  $-\infty$ , in which case the limits technically don't exist.

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Here is our first practical usage of limits: Consider the examples below.

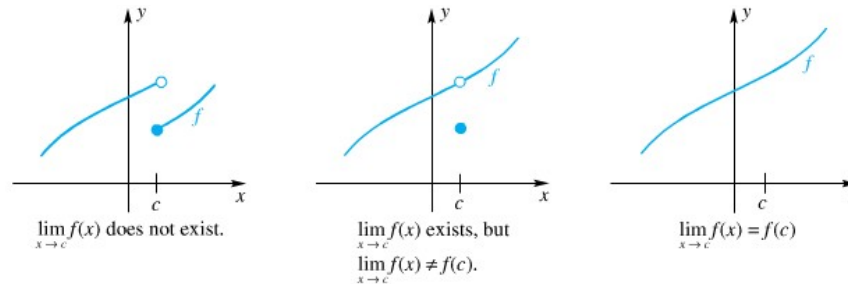


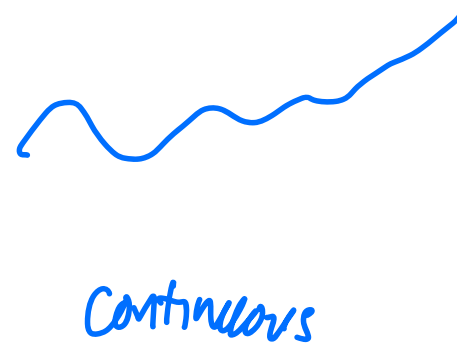
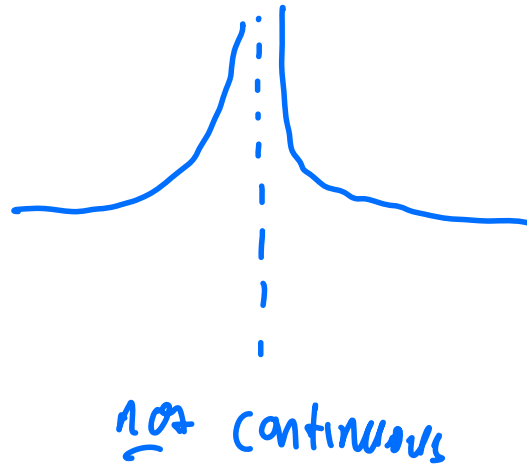
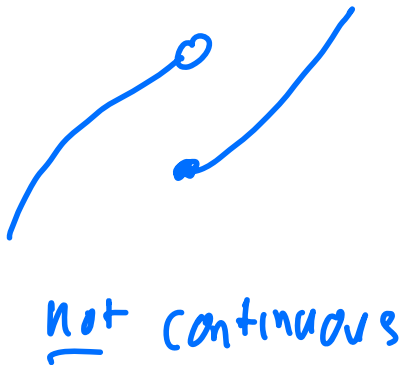
Figure 1

These examples are conceptually different. In particular something is “nicer” in the third plot.

# Continuity

D08-S03(a)

The notion of *continuity* of functions is something graphically simple to motivate: Functions are continuous at points where their graphs can be drawn without lifting your writing utensil.



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D08-S03(b)

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## Definition

A function  $f(x)$  is continuous at a point  $x = c$  if

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# Continuity

D08-S03(c)

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## Definition

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This definition means that all three of the following must be true:

1.  $f(c)$  must exist
2.  $L = \lim_{x \rightarrow c} f(x)$  must exist
3.  $L = f(c)$

Ex:  $y = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Functions cannot be continuous where they are undefined, or have vertical asymptotes.

If  $f$  is not continuous at a point, then it is *discontinuous* there.

## Example

Consider the function  $f(x) = \frac{9-x^2}{x-3}$ . Show that  $f$  is not continuous at  $x = 3$ , and determine how  $f$  should be defined at  $x = 3$  to make it continuous.

(Ans:  $f(3) = -6$ )

$f(3)$  is undefined (0/0), so it's not continuous at  $x = 3$ .

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{9-x^2}{x-3} = \lim_{x \rightarrow 3} \frac{(3-x)(3+x)}{(x-3)}$$

$$= -\lim_{x \rightarrow 3} (3+x) = -6$$

Want  $f(3) = \lim_{x \rightarrow 3} f(x) = -6$ , so define  $f(3) = -6$ , then  $f$  is continuous at  $x = 3$

# Types of discontinuities

D08-S05(a)

In the previous example,  $f$  was discontinuous at  $x = 3$ , but a simple redefinition of  $f$  removed that problem.

Such locations of discontinuity, those that can be resolved by simple redefinition of a single function value, are called removable discontinuities.

at a single point

Ex:  $y(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$x=0$  is not a removable discontinuity



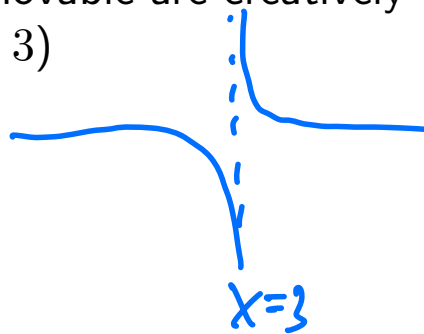
# Types of discontinuities

D08-S05(b)

In the previous example,  $f$  was discontinuous at  $x = 3$ , but a simple redefinition of  $f$  removed that problem.

Such locations of discontinuity, those that can be resolved by simple redefinition of a single function value, are called removable discontinuities.

Discontinuities that are not removable are creatively termed nonremovable discontinuities.  
(E.g.,  $f(x) = 1/(x - 3)$  at  $x = 3$ )



Recall that for “nice” functions, limits can be evaluated simply by evaluating the function. (E.g., polynomials, rational functions on their domains, trigonometric functions on their domains.)

This implies the following:

## Theorem

*Polynomial, rational, and trigonometric functions are all continuous at any point in their domain.*

(The absolute value function is continuous at every point. The  $n$ th root function for odd  $n$  is continuous at every point; if  $n$  is even, the  $n$ th root is continuous at every positive point.)


$$f(x) = |x|$$

Recall that limits are well-defined under standard arithmetic operations. This implies that the same is true for continuity of functions:

## Theorem

*If  $f$  and  $g$  are continuous functions at  $x = c$ , then  $f + g$ ,  $f - g$ ,  $f \cdot g$  are all continuous at  $x = c$ , and  $f/g$  is continuous at  $x = c$  if  $g(c) \neq 0$ .*

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You've seen that limits can be easily evaluated so long as something "wrong" doesn't happen when evaluating the function.

Likewise, functions are continuous at points where nothing "wrong" happens when evaluating the function.

## Example

Determine the points where  $f(x) = \frac{x - \sqrt{x}}{x^2 - 4}$  is continuous.

(Ans: For all  $x > 0$ , except  $x \neq 2$ )

1.)  $\sqrt{x}$  undefined for  $x < 0$

2.)  $x^2 - 4 = 0$  when  $x = \pm 2$

3.)  $\sqrt{x}$  is not continuous at  $x=0$  (limit doesn't exist)

$f$  is continuous everywhere else: for  $x > 0$  but not  $x \neq 2$ .

## Example

Determine the points where  $f(x) = \frac{1-\cos x}{x^2-x}$  is discontinuous.

(Ans: At  $x = 0$  (removable),  $x = 1$  (nonremovable))

"Problem" points:  $x^2 - x = 0 \Rightarrow x = 0, 1$

$f$  is discontinuous at  $x=0, x=1$ .

"Flavor" of discontinuity:  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1-\cos x}{x(x-1)} = \left( \lim_{x \rightarrow 0} \frac{1-\cos x}{x} \right) \cdot \left( \lim_{x \rightarrow 0} \frac{1}{x-1} \right)$ .

$\Rightarrow x=0$  is a removable discontinuity.

$$\left( \lim_{x \rightarrow 0} \frac{1}{x-1} \right) = 0 \cdot (-1) = 0$$

One-sided continuity follows just as one-sided limits.

## Definition

The function  $f(x)$  is right-continuous at  $x = c$  if  $\lim_{x \rightarrow c^+} f(x) = f(c)$ .

The function  $f(x)$  is left-continuous at  $x = c$  if  $\lim_{x \rightarrow c^-} f(x) = f(c)$ .

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Continuity at a single point generalizes to continuity on a set of points:

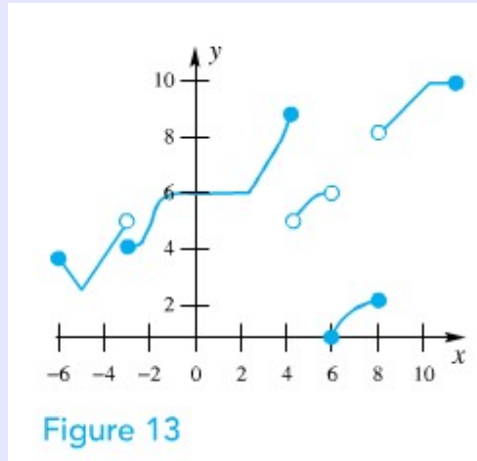
A function  $f(x)$  is continuous on an open interval  $(a, b)$  if it's continuous at every point  $x$  inside  $(a, b)$ .

A function  $f(x)$  is continuous on a closed interval  $[a, b]$  if it's continuous at every point  $x$  inside  $(a, b)$  and left-continuous at  $x = b$  and right-continuous at  $x = a$ .



## Example

Describe where the function  $g(x)$ , plotted below, is discontinuous. At these points, determine if  $g$  is left-continuous, right-continuous, or neither.



# The Intermediate Value Theorem

D08-S11(a)

Here is a very nice result that leverages continuity: Suppose  $f(0) = 0$  and  $f(1) = 2$ , and that  $f$  is continuous on  $[0, 1]$ .

This means that I should be able to draw the graph of  $f$  that connects  $(0, 0)$  and  $(1, 2)$ .

Because of this, there must be some value between  $x = 0$  and  $x = 1$  where  $f$  reaches the intermediate value 1.

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## Theorem (Intermediate Value Theorem)

*Suppose  $f$  is continuous on the interval  $[a, b]$ , and let  $W$  be a number between the values  $f(a)$  and  $f(b)$ . Then there is at least one number  $c$  on the interval  $[a, b]$  such that  $f(c) = W$ .*

## Example (t)

Show that there is at least one solution to the equation  $x - \cos x = 0$  on the interval  $[0, \pi/2]$ .

# References I

D08-S13(a)



Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall.  
ISBN: 978-0-13-142924-6.