Math 1210: Calculus I
Continuity of functions

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 1.6

Limits again

We've discussed motivation, definitions, and computations for limits, in particular for the statements,

$$\lim_{x \to c} f(x) = L, \qquad \lim_{x \to c^-} f(x) = L_-,$$

where c can be a real number or $\pm\infty$.

If L or L_- are real numbers, then the limit exists.

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Here is our first practical usage of limits: Consider the examples below.

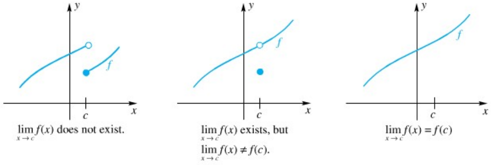

Figure 1

These examples are conceptually different. In particular something is "nicer" in the third plot.

Continuity

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Functions are continuous at points where their graphs can be drawn without lifting your writing utensil.

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This definition means that all three of the following must be true:

1. $f(c)$ must exist
2. $L = \lim_{x \to c} f(x)$ must exist
3. $L = f(c)$

Functions cannot be continuous where they are undefined, or have vertical asymptotes.

If f is not continuous at a point, then it is *discontinuous* there.

An example

Example

Consider the function $f(x) = \frac{9-x^2}{x-3}$. Show that f is not continuous at $x = 3$, and determine how f should be defined at $x = 3$ to make it continuous.
(Ans: $f(3) = -6$)

Types of discontiuities

In the previous example, f was discontinuous at $x = 3$, but a simple redefinition of f removed that problem.

Such locations of discontinuity, those that can be resolved by simple redefinition of a single function value, are called removable discontinuities.

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Discontinuities that are not removable are creatively termed nonremovable discontinuities. (E.g., $f(x) = 1/(x - 3)$ at $x = 3$)

Continuity of different functions

Recall that for "nice" functions, limits can be evaluated simply be evaluating the function.
(E.g., polynomials, rational functions on their domains, trigonometric functions on their domains.)

This implies the following:

Theorem

Polynomial, rational, and trigonometric functions are all continuous at any point in their domain.

(The absolute value function is continuous at every point. The nth root function for odd n is continuous at every point; if n is even, the nth root is continuous at every positive point.)

Continuity under operations

Recall that limits are well-defined under standard arithmetic operations. This implies that the same is true for continuity of functions:

Theorem

If f and g are continuous functions at $x = c$, then $f + g$, $f - g$, $f \cdot g$ are all continuous at $x = c$, and f/g is continuous at $x = c$ if $g(c) \neq 0$.

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You've seen that limits can be esaily evaluated so long as something "wrong" doesn't happen when evaluating the function.
Likewise, functions are continuous at points where nothing "wrong" happens when evaluating the function.

Example

Determine the points where $f(x) = \frac{x-\sqrt{x}}{x^2-4}$ is continuous.
(Ans: For all $x > 0$, except $x \neq 2$)

More examples

Example

Determine the points where $f(x) = \frac{1-\cos x}{x^2-x}$ is discontinuous.
(Ans: At $x = 0$ (removable), $x = 1$ (nonremovable))

Continuity on an interval and one-sided continuity

One-sided continuity follows just as one-sided limits.

Definition

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Continuity at a single point generalizes to continuity on a set of points:
A function $f(x)$ is continuous on an open interval (a, b) if it's continuous at every point x inside (a, b).

A function $f(x)$ is continuous on a closed interval $[a, b]$ if it's continuous at every point x inside (a, b) <u>and</u> left-continuous at $x = b$ and right-continuous at $x = a$.

Example

Describe where the function $g(x)$, plotted below, is discontinuous. At these points, determine if g is left-continuous, right-continuous, or neither.

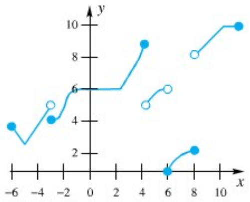

Figure 13

The Intermediate Value Theorem

Here is a very nice result that leverages continuity: Suppose $f(0) = 0$ and $f(1) = 2$, and that f is continuous on $[0, 1]$.

This means that I should be able to draw the graph of f that connects $(0, 0)$ and $(1, 2)$.

Because of this, there must be some value between $x = 0$ and $x = 1$ where f reaches the intermediate value 1.

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Theorem (Intermediate Value Theorem)

Suppose f is continuous on the interval $[a, b]$, and let W be a number between the values $f(a)$ and $f(b)$. Then there is at least one number c on the interval $[a, b]$ such that $f(c) = W$.

Example (t)

Show that there is at least one solution to the equation $x - \cos x = 0$ on the interval $[0, \pi/2]$.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.