Math 1210: Calculus I
Warmup to derivatives

Department of Mathematics, University of Utah

Spring 2025

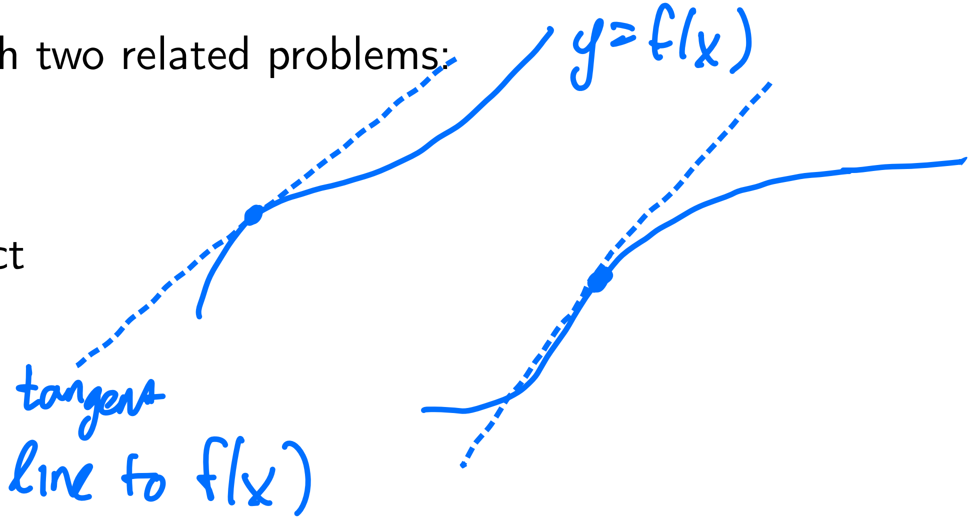
Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.1

Two focal problems

Limits will help us articulate two focal problems that will motivate perhaps the most important topic in this course: the derivative.

We'll ease into this by motivating the concept with two related problems:

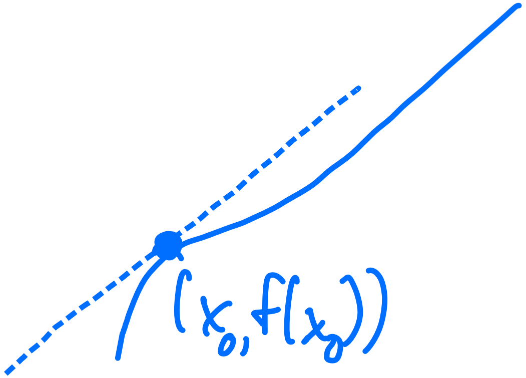
- Computing the tangent line to a curve/graph
- Computing *instantaneous* velocity of an object



The tangent line

This first problem considers computing the line that is *tangent* to a curve at a particular point.

find an equation



Need: 1.) point on line ✓ $(x_0, f(x_0))$

2.) Slope of line

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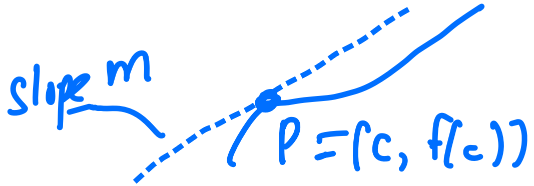
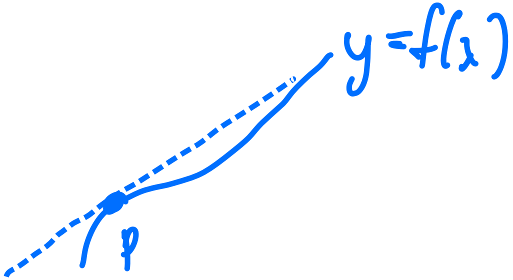


Let's start by assuming the curve is the graph of a function $y = f(x)$.
Evidently, there are two things we need in order to compute this line:

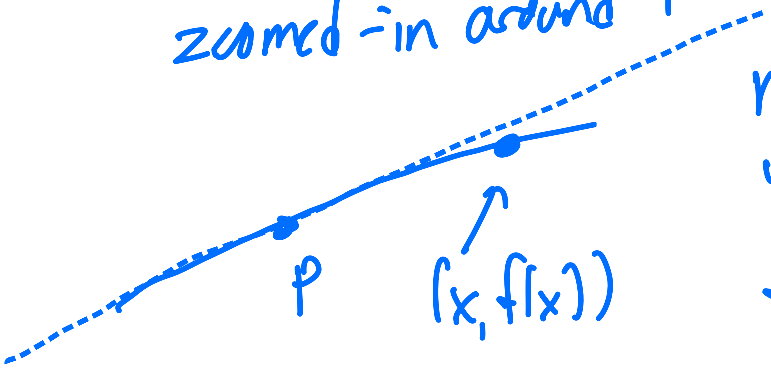
- The point P. We'll let this be the point with x coordinate equal to a constant c.
  Hence, P is the point $(c, f(c))$.
- The slope of this line. (The slope of the tangent line at the point P.) $m = ?$

Computing the slope: secant to tangent lines

We need two points to compute a slope. The point $P = (c, f(c))$ is one point. What about the second point?



zoomed-in around P



near P: $f(x)$ is "very close" to tangent line.

Idea: pick a point x "close" to $x = c$, use $(x, f(x))$ as (an approx. to) the second point

Computing the slope: secant to tangent lines

We need two points to compute a slope. The point $P = (c, f(c))$ is one point.
What about the second point?

The tangent line *approximates* the curve $y = f(x)$.
Equivalently, an approximation to the tangent line is the curve $y = f(x)$ itself. By using any other point on the graph of $y = f(x)$, we can compute a secant line through P.

The role of limits

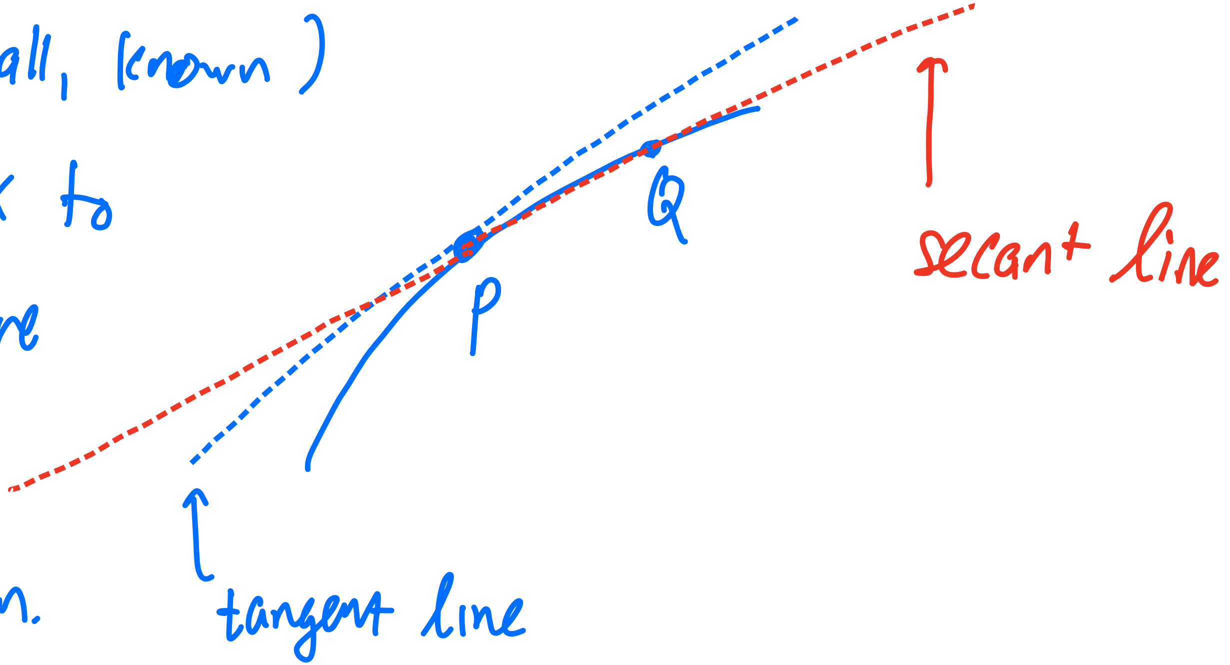
The slope of the secant line through $P = (c, f(c))$ and $Q = (c+h, f(c+h))$ is,

$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h}.$$

(h small, known)

m_{sec} is an approx to Slope of tangent line

Small $h \Rightarrow$ better approximation.



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The slope of the secant line through $P = (c, f(c))$ and $Q = (c + h, f(c + h))$ is,

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Of course, the slope of this secant line is a better approximation to the tangent line slope when h is small.

We cannot take $h = 0$, but we know how to take the *limit* as h goes to 0.

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Definition (Slope of the tangent line)

Let $P = (c, f(c))$ be a point on the graph of $y = f(x)$. The slope of the tangent line to the graph at P is,

$$m_{\tan} := \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

assuming this limit exists.

$$= \lim_{A \to 0} \frac{f(c+A) - f(c)}{A}$$

The tangent line

With a point on the tangent line $P = (c, f(c))$, and its slope m_{tan}, we can immediately identify the tangent line itself.

Definition (Tangent line to a graph)

Let $P = (c, f(c))$ be a point on the graph of $y = f(x)$. The tangent line to the graph at P is the set of points (x, y) satisfying,

$$y - f(c) = m_{\text{tan}}(x - c)$$

Example

Find the slope of the tangent line to the curve $y = x^2$ at the point $(2, 4)$. Compute the equation of the corresponding tangent line.
(Ans: Slope 4, equation $y = 4x - 4$.)

$f(x) = x^2 \quad c = 2$

$4 = f(2)$

$P = (2, 4)$

$(c+h)^2$

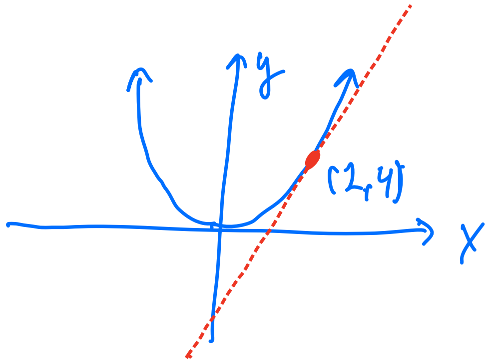
$$m_{tan} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - (2)^2}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = 4$$

Compute eqn of line through $P = (2,4)$, with slope $m_{tan} = 4$.

$$y - 4 = 4(x-2)$$

$$\Rightarrow y = 4x - 4$$



Example

Find the equation of the tangent line to the curve $y = 1/x$ at the point $\left(2, \frac{1}{2}\right)$.
(Ans: Slope $-\frac{1}{4}$, equation $y = -\frac{x}{4} + 1$.)

(Exercise on your own)

Example

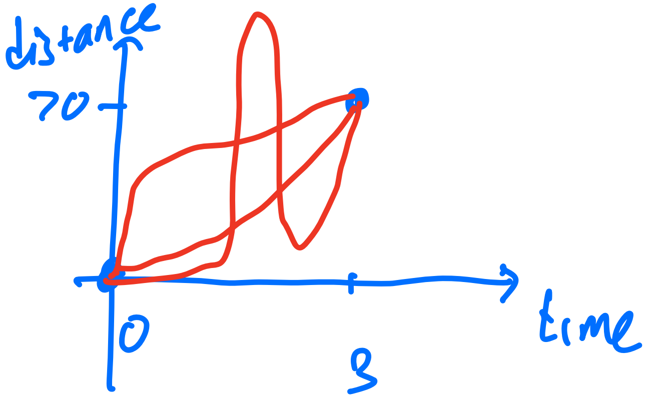
Compute the slopes of the tangent lines to the curve $y = 2x^2 - 2$ at the points with x coordinates $-1, \frac{1}{2}, 2$, and 3.
(Ans: Slopes $-4, 2, 8$, and 12, respectively)

(Exercise on your own)

The second problem: instantaneous velocity

Average velocity is an intuitive concept:
If I drive a car 70 kilometers over the course of 3 hours, my average velocity is $\frac{70}{3}$ kph (kilometers per hour).

But there are *several* different ways I can achieve this 3-hour outcome.



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But there are *several* different ways I can achieve this 3-hour outcome.

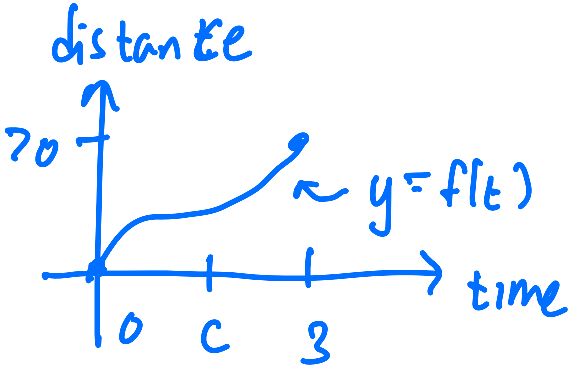
And my *instantaneous* velocity, say at one hour into the trip, can be essentially anything.

Instantaneous velocity

Instantaneous velocity is essentially the same computation we've just done with tangent lines:

Suppose $y = f(t)$ describes my position (y) as a function of time (t).
The average velocity over the time interval $[c, c+h]$ is given by,

$$\frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h}$$



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It is perhaps now not surprising to define *instantaneous* velocity at $t = c$ as the limit of this expression as h vanishes.

Definition

Let $y = f(t)$ describe the position of an object as a function of time t. The instantaneous velocity of the object at time $t = c$ is,

$$v = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Another example

Example

A particle moves in a line, traveling a total distance $s(t)$ as a function of time t given by $s(t) = \sqrt{3t + 2}$ for $t \geqslant 0$. Compute the instantaneous velocity of the particle as a function of time. (Ans: velocity $\frac{3}{2\sqrt{3t+2}}$.)

$$v(t) = \lim_{h \to 0} \frac{\cancel{f}^{\,s}(t+h) - \cancel{f}^{\,s}(t)}{h}$$ (instantaneous velocity at time t)

$$= \lim_{h \to 0} \frac{\sqrt{3(t+h)+2} - \sqrt{3t+2}}{h}$$ (Direct sub. fails)

(multiply top & bottom by $\sqrt{3(t+h)+2} + \sqrt{3t+2}$)

$$= \lim_{h \to 0} \frac{(3(t+h)+2) - (3t+2)}{h\left(\sqrt{3(t+h)+2} + \sqrt{3t+2}\right)}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3(t+h)+2} + \sqrt{3t+2}\right)}$$

$$= \frac{3}{\sqrt{3t+2} + \sqrt{3t+2}} = \frac{3}{2\sqrt{3t+2}}$$ (instantaneous velocity)

substitution

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.