Math 1210: Calculus I
Warmup to derivatives

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.1

Two focal problems

Limits will help us articulate two focal problems that will motivate perhaps the most important topic in this course: the derivative.

We'll ease into this by motivating the concept with two related problems:

- Computing the tangent line to a curve/graph
- Computing *instantaneous* velocity of an object

The tangent line

This first problem considers computing the line that is *tangent* to a curve at a particular point.

The tangent line

This first problem considers computing the line that is *tangent* to a curve at a particular point.

Let's start by assuming the curve is the graph of a function $y = f(x)$.
Evidently, there are two things we need in order to compute this line:

- The point P. We'll let this be the point with x coordinate equal to a constant c. Hence, P is the point $(c, f(c))$.
- The slope of this line. (The slope of the tangent line at the point P.)

Computing the slope: secant to tangent lines

We need two points to compute a slope. The point $P = (c, f(c))$ is one point.
What about the second point?

Computing the slope: secant to tangent lines

We need two points to compute a slope. The point $P = (c, f(c))$ is one point.
What about the second point?

The tangent line *approximates* the curve $y = f(x)$.
Equivalently, an approximation to the tangent line is the curve $y = f(x)$ itself. By using any other point on the graph of $y = f(x)$, we can compute a secant line through P.

The role of limits

The slope of the secant line through $P = (c, f(c))$ and $Q = (c+h, f(c+h))$ is,

$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{c + h - c} = \frac{f(c+h) - f(c)}{h}.$$

The role of limits

The slope of the secant line through $P = (c, f(c))$ and $Q = (c + h, f(c + h))$ is,

$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{c + h - c} = \frac{f(c+h) - f(c)}{h}.$$

Of course, the slope of this secant line is a better approximation to the tangent line slope when h is small.

We cannot take $h = 0$, but we know how to take the *limit* as h goes to 0.

The role of limits

The slope of the secant line through $P = (c, f(c))$ and $Q = (c+h, f(c+h))$ is,

$$m_{\text{sec}} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h)-f(c)}{h}.$$

Of course, the slope of this secant line is a better approximation to the tangent line slope when h is small.

We cannot take $h = 0$, but we know how to take the *limit* as h goes to 0.

Definition (Slope of the tangent line)

Let $P = (c, f(c))$ be a point on the graph of $y = f(x)$. The slope of the tangent line to the graph at P is,

$$m_{\text{tan}} := \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

assuming this limit exists.

The tangent line

With a point on the tangent line $P = (c, f(c))$, and its slope $m_{\tan}$, we can immediately identify the tangent line itself.

Definition (Tangent line to a graph)

Let $P = (c, f(c))$ be a point on the graph of $y = f(x)$. The tangent line to the graph at P is the set of points (x, y) satisfying,

$$y - f(c) = m_{\tan}(x - c)$$

Example

Find the slope of the tangent line to the curve $y = x^2$ at the point $(2, 4)$. Compute the equation of the corresponding tangent line.
(Ans: Slope 4, equation $y = 4x - 4$.)

Example

Find the equation of the tangent line to the curve $y = 1/x$ at the point $\left(2, \frac{1}{2}\right)$.
(Ans: Slope $-\frac{1}{4}$, equation $y = -\frac{x}{4} + 1$.)

Example

Compute the slopes of the tangent lines to the curve $y = 2x^2 - 2$ at the points with x coordinates $-1, \frac{1}{2}, 2$, and 3.
(Ans: Slopes $-4, 2, 8$, and 12, respectively)

The second problem: instantaneous velocity

Average velocity is an intuitive concept:
If I drive a car 70 kilometers over the course of 3 hours, my average velocity is $\frac{70}{3}$ kph (kilometers per hour).

But there are *several* different ways I can achieve this 3-hour outcome.

The second problem: instantaneous velocity

Average velocity is an intuitive concept:
If I drive a car 70 kilometers over the course of 3 hours, my average velocity is $\frac{70}{3}$ kph (kilometers per hour).

But there are *several* different ways I can achieve this 3-hour outcome.

And my *instantaneous* velocity, say at one hour into the trip, can be essentially anything.

Instantaneous velocity

Instantaneous velocity is essentially the same computation we've just done with tangent lines:

Suppose $y = f(t)$ describes my position (y) as a function of time (t).
The average velocity over the time interval $[c, c+h]$ is given by,

$$\frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h}$$

Instantaneous velocity

Instantaneous velocity is essentially the same computation we've just done with tangent lines:

Suppose $y = f(t)$ describes my position (y) as a function of time (t).
The average velocity over the time interval $[c, c+h]$ is given by,

$$\frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h}$$

It is perhaps now not surprising to define *instantaneous* velocity at $t = c$ as the limit of this expression as h vanishes.

Definition

Let $y = f(t)$ describe the position of an object as a function of time t. The instantaneous velocity of the object at time $t = c$ is,

$$v = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Another example

Example

A particle moves in a line, traveling a total distance $s(t)$ as a function of time t given by $s(t) = \sqrt{3t-2}$ for $t \geqslant 0$. Compute the instantaneous velocity of the particle as a function of time. (Ans: velocity $\frac{3}{2\sqrt{3t-2}}$.)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.