Math 1210: Calculus I
The derivative

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.2

Reminder: Two focal problems

We've considered two related problems:

- Computing the tangent line to a curve/graph

- Computing *instantaneous* velocity of an object

The fundamental challenge for both of these problems involves computing a single particular limit.

Reminder: Two focal problems

We've considered two related problems:

- Computing the tangent line to a curve/graph
- Computing *instantaneous* velocity of an object

The fundamental challenge for both of these problems involves computing a single particular limit.

Definition

Let $f(x)$ be a given function. The "derived" function,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

is called the **derivative** of the function f, defined at values of x where the limit exists.
" f' " is read "f prime".
f' is the derivative of f.

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Putting some things together:

- Fixing x, then $f'(x)$ is the slope of the tangent line to the graph of f at the point x.
- Fixing x, then $f'(x)$ is the instantaneous rate of change ("velocity") of $f(x)$.

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Putting some things together:

- Fixing x, then $f'(x)$ is the slope of the tangent line to the graph of f at the point x.
- Fixing x, then $f'(x)$ is the instantaneous rate of change ("velocity") of $f(x)$.
- $f(x)$ is a function of x. $f'(x)$ is another function of x.
- The domain of f' is the set of values x at which the limit above exists.
- If f is not defined at x, then f' cannot be defined at x.

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Putting some things together:

- Fixing x, then $f'(x)$ is the slope of the tangent line to the graph of f at the point x.
- Fixing x, then $f'(x)$ is the instantaneous rate of change (“velocity”) of $f(x)$.
- $f(x)$ is a function of x. $f'(x)$ is another function of x.
- The domain of f' is the set of values x at which the limit above exists.
- If f is not defined at x, then f' cannot be defined at x.
- $f'(3)$ means the value of the function $f'(x)$ evaluated at $x = 3$.
- $f(3)$ and $f'(3)$ can have very different values!

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Putting some things together:

- Fixing x, then $f'(x)$ is the slope of the tangent line to the graph of f at the point x.
- Fixing x, then $f'(x)$ is the instantaneous rate of change ("velocity") of $f(x)$.
- $f(x)$ is a function of x. $f'(x)$ is another function of x.
- The domain of f' is the set of values x at which the limit above exists.
- If f is not defined at x, then f' cannot be defined at x.
- $f'(3)$ means the value of the function $f'(x)$ evaluated at $x = 3$.
- $f(3)$ and $f'(3)$ can have very different values!
- The operation of taking the derivative of a function is "differentiation".
- If f has a derivative, we say that f is "differentiable".
- We are dipping our toes into "differential calculus": the study of derivatives.

Example

Compute $f'(x)$ if $f(x) = x^2 + 4$.
(Ans: $f'(x) = 2x$)

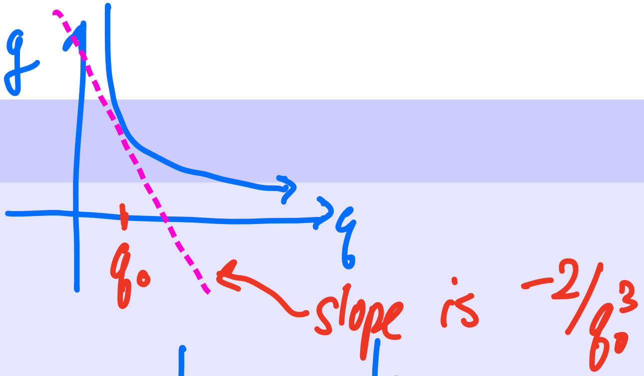
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 + 4 = x^2 + 4 + 2xh + h^2$$

$$f(x) = x^2 + 4$$

$$f'(x) = \lim_{h\to 0} \frac{x^2 + 4 + 2xh + h^2 - x^2 - 4}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = 2x$$

Derivative examples



Example

Compute $g'(q)$ if $g(q) = \frac{1}{q^2}$.
(Ans: $g'(q) = -2/q^3$)

$$g'(q) = \lim_{h\to 0} \frac{g(q+h) - g(q)}{h} = \lim_{h\to 0} \frac{\frac{1}{(q+h)^2} - \frac{1}{q^2}}{h} = \lim_{h\to 0} \frac{\frac{q^2 - (q+h)^2}{q^2(q+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{q^2 - q^2 - 2qh - h^2}{q^2(q+h)^2}}{h} = \lim_{h\to 0} \frac{-2q - h}{q^2(q+h)^2} \overset{h=0}{=} \frac{-2q - 0}{q^2 \cdot q^2} = \frac{-2}{q^3}$$

Derivative examples

Example

Compute $f'(x)$ if $f(x) = \sin x$.
(Ans: $f'(x) = \cos x$)

(Exercise on your own)

Ex. Compute $f'(x)$ for $f(x) = |x|$

$$= \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

For $x > 0$: $f(x) = x$

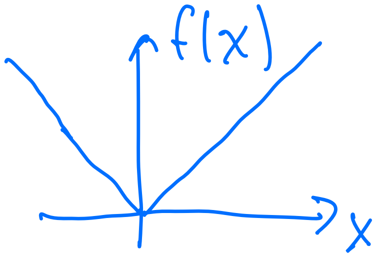
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h}$$

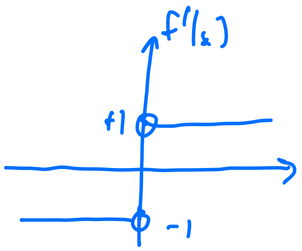
$$= \underline{1}$$

For $x < 0$: $f(x) = -x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-x \cancel{+h} - h - (-x)}{h}$$

$$= \underline{-1}$$





$f'(0)$ is undefined

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

when $x = 0$

Differentiability and continuity

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If $f'(x)$ exists, then from this definition, note that

- $f(x)$ must exist
- We must have $\lim_{h \to 0} f(x+h) = f(x)$
  (if not, the limit doesn't exist because the denominator vanishes but the numerator does not)

Differentiability and continuity

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If $f'(x)$ exists, then from this definition, note that

- $f(x)$ must exist
- We must have $\lim_{h\to 0} f(x+h) = f(x)$
  (if not, the limit doesn't exist because the denominator vanishes but the numerator does not)

Theorem

If a function f is differentiable at x, then it is continuous at x.

Differentiability and continuity

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If $f'(x)$ exists, then from this definition, note that

- $f(x)$ must exist
- We must have $\lim_{h \to 0} f(x+h) = f(x)$
  (if not, the limit doesn't exist because the denominator vanishes but the numerator does not)

Theorem

If a function f is differentiable at x, then it is continuous at x.

The converse is <u>not</u> true: If f is continuous at x, we don't know if f is differentiable there.

An equivalent derivative definition

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The above has an equivalent definition under the substitution $x = c + h$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

An equivalent derivative definition

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The above has an equivalent definition under the substitution $x = c + h$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Example

Compute $f'(x)$ if $f(x) = x^2 + 4$ using the second definition above.

Increments

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This second definition reveals some new insight: the above is a ratio of “increments”.

Increments

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This second definition reveals some new insight: the above is a ratio of "increments".

When x is close to c, then $x - c$ is an "increment" of the x value. We denote this as,

$$\Delta x = x - c, \qquad \text{Read "delta } x\text{"}$$

Increments

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This second definition reveals some new insight: the above is a ratio of "increments".

When x is close to c, then $x - c$ is an "increment" of the x value. We denote this as,

$$\Delta x = x - c, \qquad \text{Read "delta } x\text{"}$$

Similarly, the numerator is an increment of f:

$$\Delta f = f(x) - f(c).$$

Using this notation, we have,

$$f'(c) = \lim_{x \to c} \frac{\Delta f}{\Delta c}.$$

Note for all the above: Δ is not a variable, and Δ in isolation is not meaningful notation.

More derivative notation

$$f'(c) = \lim_{x \to c} \frac{\Delta f}{\Delta c}.$$

This increment notation provides new interpretation for the derivative $f'(c)$:

$f'(c)$ measures the (instantaneous) change in f relative to the instantaneous change in x at the point $x = c$.

More derivative notation

$$f'(c) = \lim_{x \to c} \frac{\Delta f}{\Delta c}.$$

This increment notation provides new interpretation for the derivative $f'(c)$:

$f'(c)$ measures the (instantaneous) change in f relative to the instantaneous change in x at the point $x = c$.

Additionally, this motivates yet another way to write the derivative:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

This notation brings another caveat: $\mathrm{d}f$ and $\mathrm{d}x$ in isolation are not meaningful (for this course).

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.