Math 1210: Calculus I
The derivative

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.2

Reminder: Two focal problems

We've considered two related problems:

- Computing the tangent line to a curve/graph

- Computing *instantaneous* velocity of an object

The fundamental challenge for both of these problems involves computing a single particular limit.

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Definition

Let $f(x)$ be a given function. The "derived" function,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

is called the **derivative** of the function f, defined at values of x where the limit exists.
" f' " is read "f prime".
f' is the derivative of f.

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Putting some things together:

- Fixing x, then $f'(x)$ is the slope of the tangent line to the graph of f at the point x.
- Fixing x, then $f'(x)$ is the instantaneous rate of change ("velocity") of $f(x)$.

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- $f(x)$ is a function of x. $f'(x)$ is another function of x.
- The domain of f' is the set of values x at which the limit above exists.
- If f is not defined at x, then f' cannot be defined at x.

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- If f is not defined at x, then f' cannot be defined at x.
- $f'(3)$ means the value of the function $f'(x)$ evaluated at $x = 3$.
- $f(3)$ and $f'(3)$ can have very different values!

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- $f'(3)$ means the value of the function $f'(x)$ evaluated at $x = 3$.
- $f(3)$ and $f'(3)$ can have very different values!
- The operation of taking the derivative of a function is "differentiation".
- If f has a derivative, we say that f is "differentiable".
- We are dipping our toes into "differential calculus": the study of derivatives.

Derivative examples

Example

Compute $f'(x)$ if $f(x) = x^2 + 4$.
(Ans: $f'(x) = 2x$)

Derivative examples

Example

Compute $g'(q)$ if $g(q) = \frac{1}{q^2}$.
(Ans: $g'(q) = -2/q^3$)

Example

Compute $f'(x)$ if $f(x) = \sin x$.
(Ans: $f'(x) = \cos x$)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If $f'(x)$ exists, then from this definition, note that

- $f(x)$ must exist
- We must have $\lim_{h \to 0} f(x+h) = f(x)$
  (if not, the limit doesn't exist because the denominator vanishes but the numerator does not)

Differentiability and continuity

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Theorem

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The converse is not true: If f is continuous at x, we don't know if f is differentiable there.

An equivalent derivative definition

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

The above has an equivalent definition under the substitution $x = c + h$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

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Example

Compute $f'(x)$ if $f(x) = x^2 + 4$ using the second definition above.

Increments

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

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Similarly, the numerator is an increment of f:

$$\Delta f = f(x) - f(c).$$

Using this notation, we have,

$$f'(c) = \lim_{x \to c} \frac{\Delta f}{\Delta c}.$$

Note for all the above: Δ is <u>not</u> a variable, and Δ in isolation is not meaningful notation.

More derivative notation

$$f'(c) = \lim_{x \to c} \frac{\Delta f}{\Delta c}.$$

This increment notation provides new interpretation for the derivative $f'(c)$:

$f'(c)$ measures the (instantaneous) change in f relative to the instantaneous change in x at the point $x = c$.

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Additionally, this motivates yet another way to write the derivative:

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

This notation brings another caveat: $\mathrm{d}f$ and $\mathrm{d}x$ in isolation are not meaningful (for this course).

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.