Math 1210: Calculus I
Rules for computing derivatives

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.3

The derivative

Given $f(x)$, then the derivative of f is another function $f'(x)$, defined as,

$$\underbrace{D_x f(x)}_{\text{book}} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

(We won't really use the notation in the first expression.)

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The definition is conceptually nice, but is unwieldy in practice.

Through the definition, we can identify rules that allow us to differentiate more easily.

Example

Show that if $g(x) = cf(x)$ for a(ny) constant c, then $g'(x) = cf'(x)$.

(Assuming f, g are differentiable)

$$g'(x) \overset{①}{=} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$\overset{②}{=} c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\overset{①}{=} c f'(x)$$

①: def'n of derivative

②: limit properties

Example

Show that if $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$.

$$h'(x) = \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} = \lim_{k \to 0} \frac{f(x+k) + g(x+k) - f(x) - g(x)}{k}$$

$$= \lim_{k \to 0} \frac{f(x+k) - f(x)}{k} + \lim_{k \to 0} \frac{g(x+k) - g(x)}{k}$$

$$= f'(x) + g'(x)$$

Differentiation is "linear"

The two previous examples can be summarized as follows:

Theorem (Linearity of differentiation)

Let $f(x)$ and $g(x)$ be differentiable functions, and let c_1 and c_2 be arbitrary constants. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c_1 f(x) + c_2 g(x)\right) = c_1 \frac{\mathrm{d}}{\mathrm{d}x} f(x) + c_2 \frac{\mathrm{d}}{\mathrm{d}x} g(x).$$

$$(c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x)$$

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Informally, one can “distribute” the derivative operation under constants and sums.

Note: by setting $c_1 = +1$ and $c_2 = -1$, one can distribute the derivative under subtraction as well.

More Examples

Example

For a(ny) fixed constant c, compute the derivative of the constant function $f(x) = c$.
(Ans: $f'(x) = 0$)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$$

$$f(1) = c$$

$$f(\pi) = c$$

$$f(1+0.001) = c$$

$$\lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0$$

More Examples

Example

Compute the derivative of the identity function $f(x) = x$.
(Ans: $f'(x) = 1$)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Example

For a(ny) fixed positive integer n, compute the derivative of $f(x) = x^n$.
(Ans: $f'(x) = nx^{n-1}$)

Some setup: recall Binomial Theorem: $(a+b)^n$

$$= c_n a^n b^0 + c_{n-1} a^{n-1} b^1 + c_{n-2} a^{n-2} b^2 + \ldots + c_1 a^1 b^{n-1} + c_0 a^0 b^n$$

For $k = 0, 1, 2 \ldots$

$$c_k = \frac{n!}{k!\,(n-k)!}, \quad c_n = 1 \text{ (because } 0! = 1\text{)}$$

$$c_{n-1} = n$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Using Binomial Theorem:

$$(x+h)^n = c_n x^n + c_{n-1} x^{n-1} \cdot h + c_{n-2} x^{n-2} h^2 + \dots + c_1 x^1 h^{n-1} + c_0 h^n$$

$$c_n = 1, \quad c_{n-1} = n$$

$$\frac{(x+h)^n - x^n}{h} = \frac{1}{\cancel{h}} \Big[\cancel{x^n} + n x^{n-1} \cancel{h} + c_{n-2} x^{n-2} h^{\cancel{2}} + \dots + c_0 h^{\overset{n-1}{\cancel{n}}} - \cancel{x^n} \Big]$$

$$= \Big[n x^{n-1} + c_{n-2} x^{n-2} h + \dots c_0 h^{n-1} \Big]$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = nx^{n-1}$$

$$\frac{d}{dx} x^n = nx^{n-1}$$

The Power Rule

We have proven the following result:

Theorem (Power Rule)

If $f(x) = x^n$ for any fixed non-negative integer n, then $f'(x) = nx^{n-1}$.
(For $n = 0$, we have $f'(x) = 0$.)

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Putting the Power Rule together with linearity of the derivative implies that polynomials can be term-by-term differentiated:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0\right) = n c_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + \ldots + c_1$$

End of Midterm Exam 1 Material

The product rule

It's tempting to hope that the rules for differentiation of sums also work for products.

Example

Let $f(x) = g(x) = x$. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)g(x)\right) \neq \left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right)$.

$$f(x)g(x) = x \cdot x = x^2$$

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}\left(x^2\right) = 2 \cdot x^1 = 2x$$

$$\left.\begin{aligned} \frac{d}{dx}f(x) &= \frac{d}{dx}x^1 = 1 \cdot x^0 = 1 \\ \frac{d}{dx}g(x) &= 1 \end{aligned}\right\} \left(\frac{d}{dx}f(x)\right)\left(\frac{d}{dx}g(x)\right) = 1$$

The product rule

It's tempting to hope that the rules for differentiation of sums also work for products.

The actual way to deal with products and derivatives is slightly more complicated than for sums.

Theorem (Product Rule)

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)g(x)\right) = g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) = f'(x)g(x) + f(x)g'(x)$$

Proof: direct from the definition of the derivative

Ex from before: $f(x) = g(x) = x$.

$f'(x) = g'(x) = 1$

product rule: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

$$= 1 \cdot x + 1 \cdot x$$

$$= 2x \checkmark$$

<u>Proof</u>: $\frac{d}{dx}(f(x)\,g(x)) = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)g(x)}{h}$

(add and subtract $f(x)g(x+h)$ in numerator)

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$\cancel{= \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h \to 0} f(x)}$$

$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h \to 0} \frac{f(x)[g(x+h) - g(x)]}{h}$$

$$= \left[\lim_{h \to 0} g(x+h)\right]\left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right]$$

$$+ \left[\lim_{h \to 0} f(x)\right]\left[\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right]$$

$$= g(x)\,f'(x) + f(x)\,g'(x) \checkmark$$

The quotient rule

Differentiation for quotients is even more symbolically complicated.

Theorem (Quotient Rule)

If $f(x)$ *and* $g(x)$ *are differentiable functions and* $g(x) \neq 0$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Example

Compute the derivative of $f(x) = (x^2+3)(x^3+4)$.
(Ans: $f'(x) = 5x^4 + 9x^2 + 8x$)

$$f(x) = \underbrace{(x^2+3)}_{g(x)}\underbrace{(x^3+4)}_{h(x)}$$

$$f'(x) = \frac{d}{dx}(g(x)h(x)) = g'(x)h(x) + g(x)h'(x)$$

$$= \left[\frac{d}{dx}(x^2+3)\right](x^3+4) + (x^2+3)\left[\frac{d}{dx}(x^3+4)\right]$$

$$= 2x(x^3+4) + (x^2+3)\cdot 3x^2$$

$$= 2x^4 + 8x + 3x^4 + 9x^2 = 5x^4 + 9x^2 + 8x$$

Ex. Compute $\frac{d}{dx}(x \cdot \sin x)$

$$= \left(\frac{d}{dx} x\right) \cdot \sin x + x \cdot \overbrace{\frac{d}{dx} \sin x}^{\cos x}$$

$$= 1 \cdot \sin x + x \cdot \cos x$$

$$= \sin x + x \cos x.$$

Example

Compute the derivative of $f(x) = \frac{x-4}{x^2+3}$.

(Ans: $f'(x) = \frac{-x^2+8x+3}{(x^2+3)^2}$)

$g(x) = x-4 \qquad h(x) = x^2+3$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{g(x)}{h(x)}\right) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$$

$$= \frac{1\cdot(x^2+3) - (x-4)(2x)}{(x^2+3)^2} = \frac{x^2+3-2x^2+8x}{(x^2+3)^2}$$

$$= \frac{-x^2+8x+3}{(x^2+3)^2}$$

Example

Let n be any integer (even negative). Show that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.
(The point: the Power Rule holds for <u>any</u> integer n, not just positive ones.)

($n \geq 0$: this is just the power rule. So only need to show this for $n < 0$).

Suppose $n < 0$: $n = -|n|$

$$f(x) = x^n = x^{-|n|} = \frac{1}{x^{|n|}} = \frac{g(x)}{h(x)}$$

$$h'(x) = |n| \cdot x^{|n|-1}, \quad \underline{g'(x) = 0}$$

$$f'(x) = \frac{d}{dx}\left(g(x)\cancel{h(x)}\right) = \frac{\overbrace{\cancel{g'(x)h(x)}}^{0} - h'(x)g(x)}{h^2(x)}$$

$$\left(g(x)/h(x)\right)$$

$$= \frac{\overbrace{-|n|}^{n} x^{|n|-1} \cdot 1}{x^{2|n|}}$$

$$= \frac{n \cdot x^{|n|-1}}{x^{2|n|}} = n x^{|n|-1-2|n|}$$

$$= n x^{\overbrace{-|n|}^{n}-1}$$

$$= n x^{n-1} \checkmark$$

Rules and rules and rules

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c_1 f(x) + c_2 g(x)\right) = c_1 f'(x) + c_2 g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) g(x)\right) = f'(x) g(x) + f(x) g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}$$

Lots of rules....

It is definitely worth memorizing these rules.
Fluency and comfort with these is *essential* moving forward.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.