Math 1210: Calculus I
Derivatives of trigonometric functions

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.4

The derivative

Given $f(x)$, then the derivative of f is another function $f'(x)$, defined as,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

We've used the definition to derive the following rules:

- Linearity: $(c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x)$
- Power Rule: $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ for any integer n. ($\frac{\mathrm{d}}{\mathrm{d}x} x^0 = 0$)
- Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

We'll now use these rules to compute derivatives of trigonometric functions.

Sine and cosine

Theorem

The derivative of the sine and cosine functions are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x.$$

(We showed the derivative of $\sin x$ previously. The $\cos x$ derivative computation is similar.)

Recall: $\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sin h - \sin x}{h}$

$= \lim_{h\to 0} \cos x \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} + \lim_{h\to 0} \sin x \underbrace{\left(\frac{\cosh - 1}{h}\right)}_{\downarrow\, 0} = \cos x.$

Sine and cosine

Theorem

The derivative of the sine and cosine functions are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x.$$

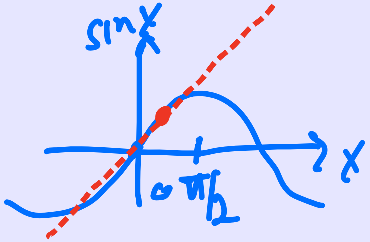
(We showed the derivative of $\sin x$ previously. The $\cos x$ derivative computation is similar.)

Example

Compute the equation of the tangent line to the graph of $y = \sin x$ at the point $x = \frac{\pi}{4}$.

(Ans: $y = \frac{\sqrt{2}}{2} x + \frac{(4-\pi)\sqrt{2}}{8}$.)

point on line: $x = \cancel{\pi/2}\ \pi/4$; $y = \sin x = \sin(\pi/4) = \sqrt{2}/2$



Slope of tangent line is $\frac{dy}{dx}$ evaluated at $x = \pi/4$ $\Big\}$ $y'(x) = \cos x \Rightarrow$ slope $= \frac{\sqrt{2}}{2}$

eqn of line: $y - y_0 = m(x - x_0)$

$$(x_0, y_0) = \left(\pi/4, \sqrt{2}/2\right)$$

$$m = \sqrt{2}/2$$

$$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(x - \pi/4\right)$$

$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \pi/4\right)$$

Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \sin x\right)$
(Ans: $2x \sin x + x^2 \cos x$)

product rule

$$\frac{d}{dx}(x^2 \sin x) = \left[\frac{d}{dx}(x^2)\right] \sin x + x^2 \frac{d}{dx}(\sin x)$$

$$= 2x \sin x + x^2 \cos x$$

Another way: $\frac{d}{dx}(x^2 \sin x) = \frac{d}{dx}[x \cdot x \sin x]$

$$= \left[\frac{d}{dx}(x)\right] \cdot x \sin x + x \cdot \frac{d}{dx}(x \sin x)$$

$$= 1 \cdot x\sin x + x\left[\frac{d}{dx}(x) \cdot \sin x + x \cdot \frac{d}{dx}(\sin x)\right]$$

$$= x \cdot \sin x + x\left[1 \cdot \sin x + x\cos x\right]$$

$$= 2x\sin x + x^2\cos x$$

Another (awful) way: $x^2\sin x = \dfrac{\sin x}{\frac{1}{x^2}} = \dfrac{\sin x}{x^{-2}}$

Example

Compute all points x where the tangent line to $y = \cos^2 x$ is horizontal.
(Ans: $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots$)

horizontal line (slope 0)

where does the tangent line have slope 0?
(derivative equals 0)

Find x such that $\frac{d}{dx} y = 0$

$y(x) = (\cos x)(\cos x)$

$$\frac{d}{dx} y = \left(\frac{d}{dx}\cos x\right)\cos x + \cos x \cdot \left(\frac{d}{dx}\cos x\right)$$

$$= -\sin x \cos x - \cos x \cdot \sin x$$

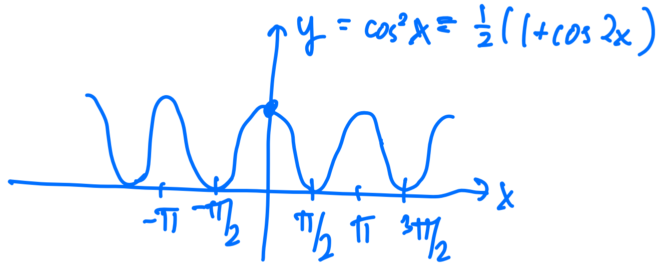
$$= -2\sin x \cos x$$

where is $-2\sin x\cos x = 0$?

Either $\sin x = 0$ or $\cos x = 0$

$x = 0, \pm\pi, \pm 2\pi, \ldots$ $\qquad$ $x = \pm\pi/2, \pm\frac{3\pi}{2}, \ldots$

Ans: $x = 0, \pm\pi/2, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \ldots$



Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x} \tan x$.
(Ans: $\sec^2 x$.)

$$\tan x = \frac{\sin x}{\cos x}$$

$$\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}[\sin x] \cdot \cos x - \frac{d}{dx}(\cos x) \cdot \sin x}{\cos^2 x}$$

$$= \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Co/secant and Co/tangent

The procedure from the previous example can be used to compute derivatives for other trigonometric functions we've encountered:

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.