Math 1210: Calculus I
Derivatives of trigonometric functions

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.4

The derivative

Given $f(x)$, then the derivative of f is another function $f'(x)$, defined as,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

We've used the definition to derive the following rules:

- Linearity: $(c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x)$
- Power Rule: $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ for any integer n. ($\frac{\mathrm{d}}{\mathrm{d}x} x^0 = 0$)
- Product rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

We'll now use these rules to compute derivatives of trigonometric functions.

Sine and cosine

Theorem

The derivative of the sine and cosine functions are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x.$$

(We showed the derivative of $\sin x$ previously. The $\cos x$ derivative computation is similar.)

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Example

Compute the equation of the tangent line to the graph of $y = \sin x$ at the point $x = \frac{\pi}{4}$.

(Ans: $y = \frac{\sqrt{2}}{2}x + \frac{(4-\pi)\sqrt{2}}{8}$.)

Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \sin x\right)$
(Ans: $2x \sin x + x^2 \cos x$)

Example

Compute all points x where the tangent line to $y = \cos^2 x$ is horizontal.
(Ans: $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots$)

Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x} \tan x$.
(Ans: $\sec^2 x$.)

Co/secant and Co/tangent

The procedure from the previous example can be used to compute derivatives for other trigonometric functions we've encountered:

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.